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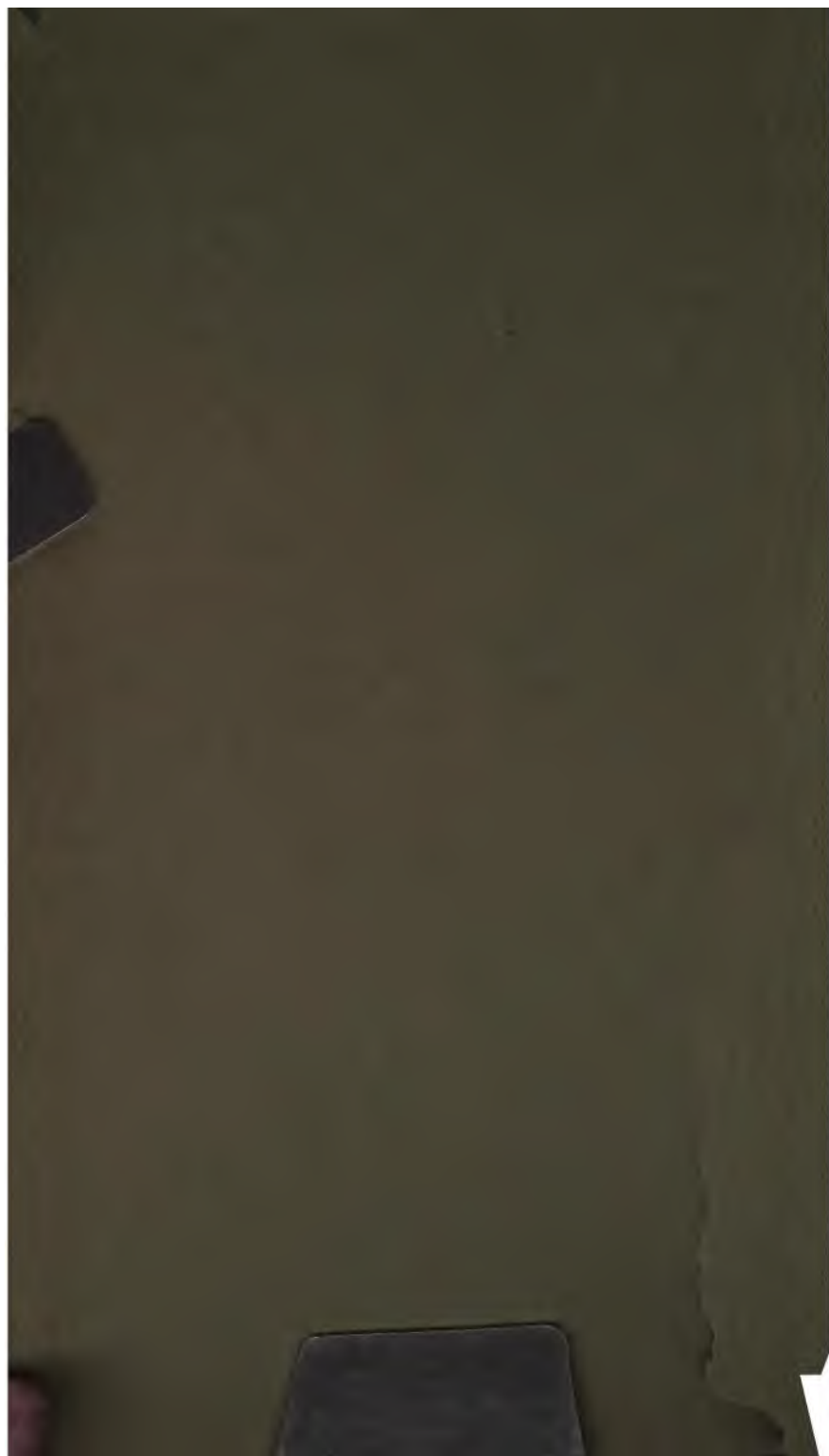
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(Bay)

1794

A
TREATISE
ON THE
ELEMENTS OF ALGEBRA;

DESIGNED FOR THE USE OF
ETON SCHOOL.

BY THE
REV. JOHN BAYLEY, M.A.

LATE FELLOW AND MATHEMATICAL LECTURER
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TO THE
REV. J. KEATE, D.D.

CANON OF WINDSOR,
AND HEAD MASTER OF ETON SCHOOL,

THE FOLLOWING PAGES ARE, WITH HIS PERMISSION,

RESPECTFULLY INSCRIBED

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AND OBEDIENT SERVANT,

THE AUTHOR.

INTRODUCTION.

THE following pages contain the substance of a course of Lectures delivered, during the last year, to my Pupils at Eton ; and the success which has followed my endeavours to smooth the first approach to Science, now induces me to publish them for the use of the School. In accordance with this object, my first intention was to treat every subject in the most familiar manner. But such a plan I afterwards abandoned as impracticable ; for, by swelling my pages, I might have become accountable for the certain and serious evil of a μέγα βιβλίον, with the uncertain prospect of any equivalent advantage. Some who might otherwise have profited by the work, would have been discouraged by its bulk. And, moreover, I considered that no book, written for a School, was ever intended to supersede the assistance of a Teacher.

I have devoted a considerable portion of the work to "Problems in Equations," and have likewise given them a place among the earliest lessons in Algebra: for, by thus dressing the nakedness of Science, and applying its principles to the transactions of common life, the Student discovers that the object of his inquiry is something more than abstruse speculation. His curiosity is awakened to try the Solution of Questions which, as they are of daily occurrence, may one day concern himself. Thus weariness is provided with relief, and leisure allured with employment. The mind that flags with encountering successive obstructions may recruit its strength by turning aside to a more inviting path; and the reluctant Student may be tempted to try his skill on a question proposed in the form of a puzzle. With a view to this end, and for a safer guidance to the learner, the Treatise is divided into Two Parts. In every instance the Reader's attention should be directed to the more simple and popular view which is taken of a subject in the *First Part*, before he proceed to prove its principles or investigate its properties in the *Second*. The more difficult proofs are placed in

the Second Part, and are generally followed by Problems illustrative of the subject. With a further view to the Reader's convenience, the same pages in both Parts refer to the same subject. Several of the Questions are selected from SAUNDERSON'S Algebra, and some from a German Treatise, by HIRSH.

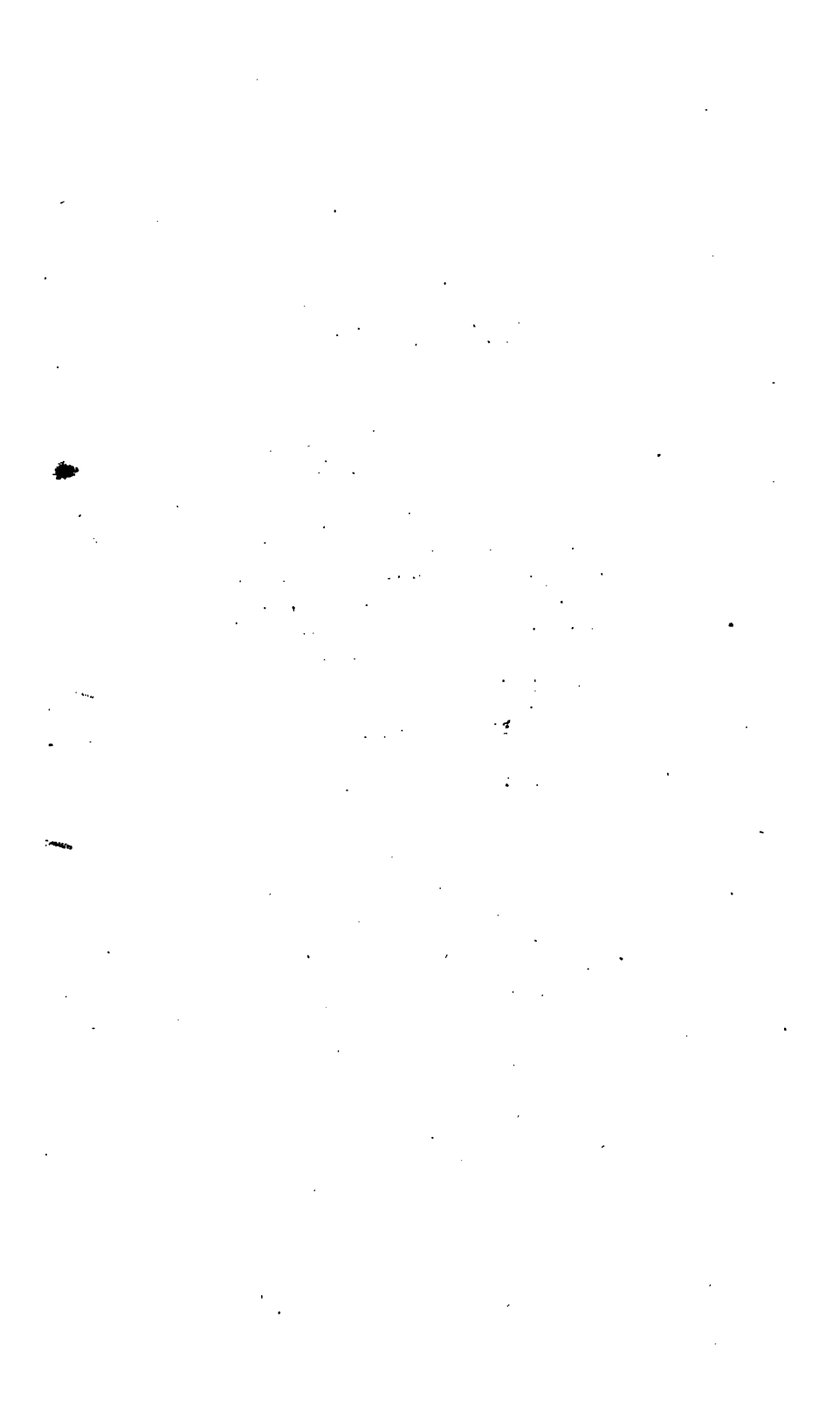
The responsibility of this work falls entirely upon myself; for in the plan and prosecution of it I neither received nor sought assistance. The propriety of such a measure I am not prepared to defend, as I may probably be found weakest where I have most trusted to my own strength; and I might have sheltered myself from the possibility of censure, had I more faithfully followed the system pursued some years ago at Trinity College by Mr. HUDSON, the present Vicar of Kendal. That name must be connected with a lively sense of gratitude in all who have enjoyed the benefit of his instruction. Numbers at this day can bear testimony to the *opportunities* of improvement afforded them under his guidance; and for myself I may venture to affirm that my obligations to the world

would be easily discharged if I could once acquit myself of the debt which has been long owing to that gentleman as a Tutor and a Friend.

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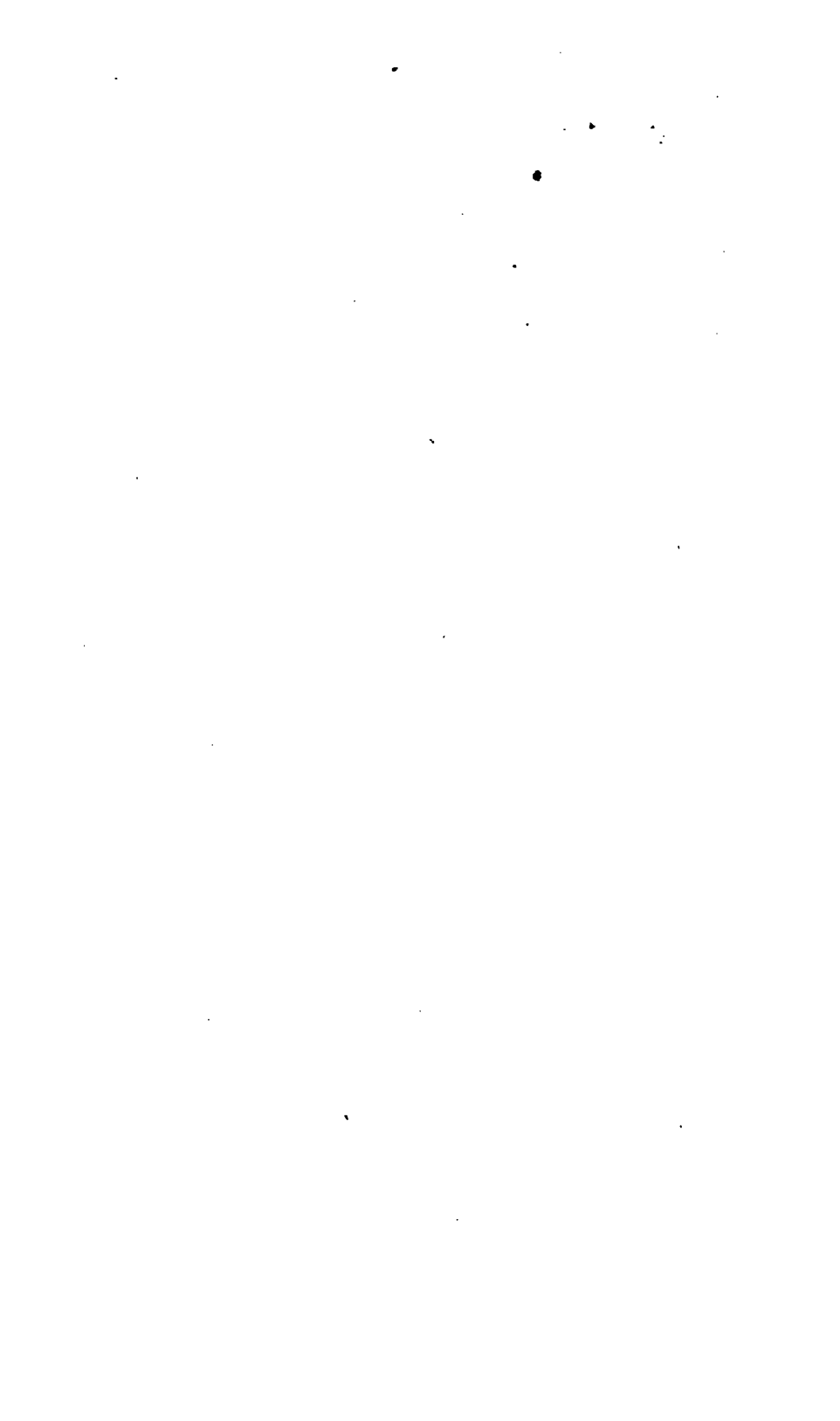
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PART I.







PART I.

ALGEBRA.

Definition.

WHATEVER admits of increase or diminution, may be called "Magnitude" or "Quantity."

Mathematics is the science which investigates the means of measuring Quantity. This may be done, either by Geometry; or by that operation of numbers which we term "Arithmetic;" or by adopting that more general instrument of computation which is called "Algebra," and has sometimes been termed "Universal Arithmetic."

Arithmetic, therefore, may be defined "the science which represents the relations of magnitudes by means of *numbers*;" and Algebra, as "the science which represents the same relations by means of *letters* or *symbols*."

Explanation of Algebraic Characters.

1. The number prefixed to an algebraic quantity is called its co-efficient, and shews how many times it is to be taken. Thus, in the expressions, $5a$, $7b$, 5 and 7 are the co-efficients of (a) and (b) respectively.

2. The number placed over an algebraic quantity is called its Index or Exponent, and denotès how often the quantity is multiplied into itself. Thus the expressions, a^2 , a^3 , a^n , denote the square, cube, n^{th} power of (a)

When two or more quantities are to be added together, they are joined to each other by this sign (+) called plus. Thus $5 + 3a + 7b^2 + 8x^3$ is read thus; five, plus three (a), plus seven (b) square, plus eight (x) cube.

When one quantity is to be subtracted from another, it

is preceded by this sign ($-$), called minus; Thus $9a^2 - 4b$ is read nine (a) square minus four (b).

Multiplication is expressed by this sign ($\times$) or by this ($\cdot$); thus $a \times b$, or, $a \cdot b$, or ab , means the product of (a) and (b).

Division is expressed by this sign $\div$; thus $a \div b$ signifies (a) divided by (b).

Equality is expressed by this sign ($=$); thus $a=b$ denotes (a) equal to (b).

Quantities joined together under an horizontal line, thus, $\overline{a+b-c}$ are said to be under a vinculum; and such quantities are to be taken together in algebraical operations: thus, $a - \overline{b+c-d}$ means that the whole of the quantities ($b+c-d$) must be subtracted from (a). For instance, let $a=12$; $b=10$, $c=5$, $d=6$; then the expression $a - \overline{b+c-d}$ becomes $12 - \overline{10+5-6} = 12 - 9 = 3$, the same result as if it were written thus $12 - 10 - 5 + 6$; if the same quantities occurred without a vinculum, the expression would be $12 - 10 + 5 - 6 = 1$.

Like Quantities are those which consist of the same letters of the alphabet, similarly combined; thus $2a$, $5a$, na , are like quantities; also $+3ax^2$ and $-7ax^2$ are like quantities; also $6x^2$, $4x^2$, mx^2 . Unlike quantities consist of different letters, or of the same letters differently involved; thus $5a$, $3x^2$, $4b$, are unlike quantities; also $5ax^2$ and $5a^2x$ are unlike quantities. Quantities are said to be positive or negative, according as a *plus* or *minus* sign is prefixed to them: thus of the quantities, $4a$, $-3b$, the first is positive, and the last is negative.

The signs of algebraic quantities are said to be like when they are all *plus* or all *minus*.

A Simple Quantity is one, in which the letters composing it are not connected by either *plus* or *minus*. Thus,

a , b^2 , xy , $\frac{x}{y}$, are simple quantities. Whereas $(a+b)$, $(a-b)$, are compound quantities.



If the index of an algebraic quantity be a fraction, it denotes some root of that quantity. Thus, $a^{\frac{1}{2}}$ (or $\sqrt[2]{a}$) denotes the square root of (a) ; $a^{\frac{1}{3}}$ (or $\sqrt[3]{a}$) denotes the cube root of (a) ; $a^{\frac{2}{3}}$ (or $\sqrt[3]{a^2}$) denotes the cube root of a^2 ; $a^{\frac{1}{4}}$ (or $\sqrt[4]{a}$) denotes the fourth root of a .

ADDITION.

ADD together the co-efficients of like quantities with like signs; take the difference between the sum of the positive and the sum of the negative co-efficients, and to that difference prefix the sign of the greater, and annex the letters of the alphabet that are common to both; thus, if I add together

$$\begin{aligned} 4a^2 + 3b \\ 9a^2 + 7b \\ 5b - 3a^2 \\ 2a^2 - 10b, \end{aligned}$$

the positive terms involving a^2 amount to $15a^2$; the negative terms involving a^2 amount to $-3a^2$; the difference is $+12a^2$; and, by a similar process, we find the aggregate of the terms involving (b) to be $+5b$; therefore the sum required is $12a^2 + 5b$.

Suppose, in this example that $a=2$, and $b=5$.

$$\begin{array}{rcl} \text{Then } 4a^2 + 3b & = & 16 + 15 = 31 \\ 9a^2 + 7b & = & 36 + 35 = 71 \\ 5b - 3a^2 & = & 25 - 12 = 13 \\ 2a^2 - 10b & = & 8 - 50 = -42 \\ \hline 12a^2 + 5b & & 115 - 42 \end{array}$$

And $12a^2 + 5b = 12 \times 4 + 25 = 73$, the same as $115 - 42$.

$$\begin{array}{rcl} \text{Again; if we add together } 5a - 2b + 3\sqrt{x} & & \\ 7a + b + 9\sqrt{x} & & \\ 15b - 2\sqrt{x} + 4a & & \\ \sqrt{x} - 8b & - & 2a \\ \hline \end{array}$$



The sum is $14a + 6b + 11\sqrt{x}$.

Suppose $a=3$; $b=5$; $x=4$, then $\sqrt{x}=2$,

$$\text{Then } 5a - 2b + 3\sqrt{x} = 15 - 10 + 6 = 11$$

$$7a + b + 9\sqrt{x} = 21 + 5 + 18 = 44$$

$$15b - 2\sqrt{x} + 4a = 75 - 4 + 12 = 83$$

$$\sqrt{x} - 8b - 2a = 2 - 40 - 6 = -44$$

$$138 - 44$$

Then $14a + 6b + 11\sqrt{x} = 42 + 30 + 22 = 94$, the same amount as $138 - 44$.

Add together $4y^2 + 6b - 10x$

$$3y^2 - x + 4b$$

$$12x - 5y^2 - 3b$$

$$2y^2 + 7b + x$$

$$6x + 5ay$$

$$-3x + 2ay$$

$$x - 6ay$$

$$2x + ay$$

$$3b + 4a - 6y$$

$$7b + 2a + y$$

$$3a - 4b + 3y$$

$$7x - 3ab$$

$$3ab - 10x$$

$$3ab - 6x$$

$$-ab + 2x$$

$$2ab + 7x$$

$$a^{\frac{1}{2}}x^2 - bx - cx^2$$

$$4cx^2 - 4a^{\frac{1}{2}}x^2 + 3bx$$

$$2cx^2 - 5bx + 91$$

$$ax^2 - 5x$$

$$-bx^2 + 4x$$

$$+cx^2 + dx$$

$$-x^2 + ax$$

$$2ax - 30$$

$$3x^2 - 2ax$$

$$5x^2 - 3\sqrt{x}$$

$$3\sqrt{x} + 10$$

$$10b^2 - 3a^2x$$

$$-b^2 + 2a^2x^2$$

$$50 + 2a^2x$$

$$a^2x^2 + 120$$

SUBTRACTION

OF Algebraic Quantities is performed by changing the signs of the subtrahend, and then proceeding as in addition.

Thus, from $5a - 6b$
take $3a + 2b$

Difference is $2a - 8b$

Suppose, for instance, that $a=12$, and $b=2$;
then the example becomes this, From $60 - 12 = 48$
Take $36 + 4 = 40$

according to the rule the difference is $24 - 16$ or 8
the same as $48 - 40$.

Ex. 2.

From $5xy + 8x - 2$
Take $3xy - 8x + 7$

Diff. $2xy + 16x - 9$

As before, substitute numbers for x and y . Thus suppose $x=3$, and $y=2$, $\therefore xy=6$.

Then $5xy + 8x - 2 = 30 + 24 - 2 = 52$

$3xy - 8x + 7 = 18 - 24 + 7 = 1$

Diff. $2xy + 16x - 9 = 12 + 48 - 9 = 51,$

the same as $52 - 1$.

Ex. 3.

From $5x^2 - 8y + 3x^2y^2$

Take $4x^2 + 8y - x^2y^2$

Diff. $x^2 - 16y + 4x^2y^2$



Ex. 4.

From $4a^2 - 3xy + 6$ Take $20 - a^2 + xy$ Diff. $5a^2 - 4xy - 14$

Ex. 5.

From $ax^2 - px + qxy^2$ Take $bx^2 - rx - cxy^2$ Diff. $a - b . x^2 + r - p . x + q + c . xy^2$

Ex. 6.

From $6x^3y + 10x^2y^2 + 13xy^3 + 17y^4$ Take $4x^3y - 2x^2y^2 + 3xy^3 - 3y^4$

Ex. 7.

From $15ax^3 - 17a^2x^2 + 15a^3x - 9y^4$ Take $17ax^3 + 21a^2x^2 - 15a^3x - 7a^4$

Ex. 8.

From $6abx + 12 - 3xy + 4xz$ Take $3abx + xz - 7 + 5xy$

MULTIPLICATION

OF like Algebraic Quantities is performed by multiplying together the co-efficients, and adding together the indices, of the quantities. To the product thus obtained prefix the negative sign, if the signs of the quantities be unlike; and the positive sign, if they be like.

Thus, $4a^2 \times 3a^2 = 12a^4$; $9b^3 y^4 \times -2b^3 y^2 = -18b^6 y^6$. This follows from Art. 2nd; for y^4 is the same as $y \times y \times y \times y$; and y^2 is the same as $y \times y$ $\therefore y^4 \times y^2 = y \times y \times y \times y \times y \times y$, or y^6 . Also, $x^m = x \times x \times x \dots$ to (m) terms; and $x^n = x \times x \times x \dots$ to (n) terms; $\therefore x^m \times x^n = x \times x \times x \dots$ to (m) terms, $\times x \times x \times x \dots$ to (n) terms $= x^{m+n}$.

If compound quantities be multiplied together, each term of the multiplicand must be multiplied by each term of the multiplier, and the sum found as in addition.

Ex. 1.

$$\begin{array}{r} 4a\ b \\ 3a^2 b^3 \\ \hline 12a^3 b^4 \end{array}$$

Ex. 2.

$$\begin{array}{r} 5x\ y^3 \\ 4x^2 y^4 \\ \hline 20x^3 y^7 \end{array}$$

Ex. 3.

$$\begin{array}{r} 6a^2 b^2 \\ -\ 4a\ b \\ \hline -24a^3 b^3 \end{array}$$

Ex. 4.

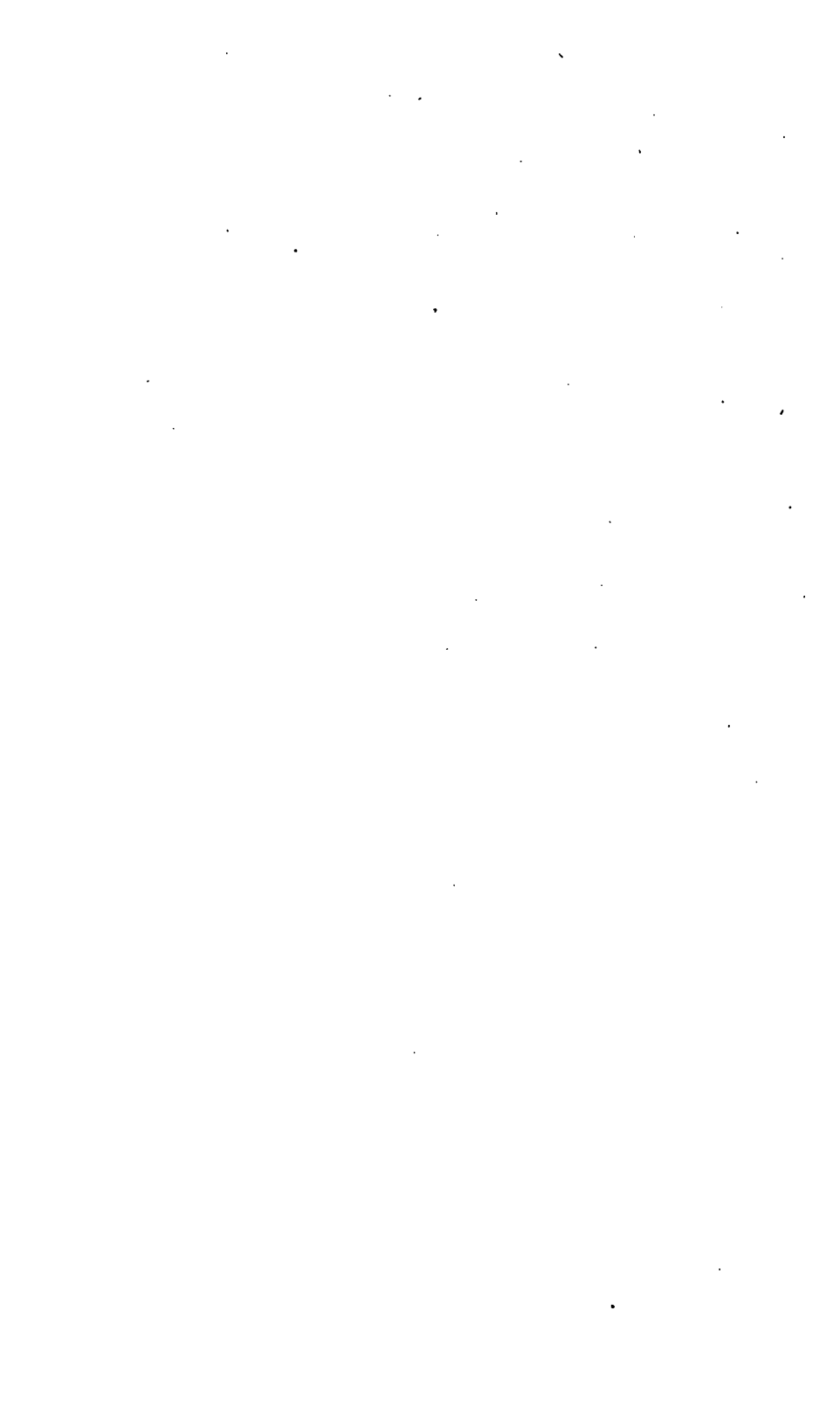
$$\begin{array}{r} -5x\ y^3 \\ -3x^2 y \\ \hline 15x^3 y^4 \end{array}$$

Ex. 5.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

Ex. 6.

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ -\ ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$





EX. 7.

$$\begin{array}{r}
 x^3 + x^2y + xy^2 + y^3 \\
 x - y \\
 \hline
 x^4 + x^3y + x^2y^2 + xy^3 \\
 -x^3y - x^2y^2 - xy^3 - y^4 \\
 \hline
 x^4 \quad * \quad * \quad * \quad -y^4 \\
 \hline
 \end{array}$$

EX. 8.

$$\begin{array}{r}
 x^3 - 2xy + y^3 \\
 x^2 + 3xy \\
 \hline
 x^4 - 2x^2y + x^2y^2 \\
 + 3x^3y - 6x^2y^2 + 3xy^3 \\
 \hline
 x^4 + x^3y - 5x^2y^2 + 3xy^3 \\
 \hline
 \end{array}$$

EX. 9.

$$\begin{array}{r}
 3a^2 + 2ab - b^2 \\
 3a^2 - 2ab + b^2 \\
 \hline
 9a^4 + 6a^3b - 3a^2b^2 \\
 -6a^3b - 4a^2b^2 + 2ab^3 \\
 + 3a^2b^3 + 2ab^3 - b^4 \\
 \hline
 9a^4 \quad * \quad -4a^2b^2 + 4ab^3 - b^4 \\
 \hline
 \end{array}$$

Multiply $x^3 + x^2y + xy^2 + y^3$
by $x^3 - y^3$

$$\begin{array}{r}
 x^5 + x^4y - xy^3 - y^5 \\
 \hline
 \end{array}$$

Multiply $2a + b$
by $2a - b$

$$\begin{array}{r}
 4a^2 - b^2 \\
 \hline
 \end{array}$$

Multiply $x^2 + 2xy + y^2$
by $x^2 - 2xy + y^2$

$$\begin{array}{r}
 x^4 - x^2y^2 + y^4 \\
 \hline
 \end{array}$$

Multiply $x^3 - 3x^2y + 3xy^2 - y^3$
by $x - y$

$$\begin{array}{r}
 x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\
 \hline
 \end{array}$$

$$\begin{array}{r} \text{Multiply } x^4 + 2x^3 + 3x^2 + 2x + 1 \\ \text{by } x^2 - 2x + 1 \end{array}$$

$$\hline x^6 - 2x^5 + 2x^4 + 1$$

$$\begin{array}{r} \text{Multiply } x^5 - x^2 + x - 5 \\ \text{by } 2x^3 + x + 1 \end{array}$$

$$\hline 2x^8 - x^4 + 2x^3 - 10x^2 - 4x - 5$$

$$\begin{array}{r} \text{Multiply } a^{n-1} + b^{n-1} \\ \text{by } a - b \end{array}$$

$$\hline a^n - a^{n-1}b + ab^{n-1} - b^n$$

Multiplication properly denotes the continued addition of one quantity (the multiplicand) to itself, as many times as there are units in another quantity called the multiplier. Still we see cases where the multiplicand and multiplier express quantities of *different kinds*, and the product a quantity *different in kind* from either. Thus if a room be 8 yards long and 5 yards broad, the floor is said to contain 8×5 (or 40) *square* yards. And if the same room be 4 yards high, its content is said to be 40×4 (or 160) *cubic* yards. But, in this case, the quantities employed denote *numbers* in reference to their *proper units*. Thus, in the question before us, 8 denotes the number of *linear units*; 40 the number of *superficial units*, and 160 the number of *solid units*. Suppose also that we wish to find the price of 20 yards of cloth at half-a-crown a yard, we do not multiply *yards* and *money* together; but we multiply 2*s.* 6*d.* by the *abstract number* 20, the number of units expressed by the number of yards.





DIVISION

Is the reverse of Multiplication, and may be performed by pursuing a method the reverse of that by which we found the product of Algebraic Quantities. Thus if $5a^3 \times 3a^3 = 15a^6$, we know that $15a^6 \div 5a^3$ must be equal to $3a^3$. Since

$$20x^4y^6 \times -4xy^2 = -80x^5y^8; \text{ it follows that } \frac{-80x^5y^8}{20x^4y^6} = -4xy^2$$

Hence it appears that Division of Simple Algebraic Quantities is performed by dividing the co-efficient of the dividend by that of the divisor, and subtracting the index of the divisor from that of the dividend.

$$1. \frac{4x^3 - 8x^2y + 4x^2z}{4x^3} = 1 - 2xy + z.$$

$$2. \frac{25a^3c - 10ac^2}{5ac} = 5a - 2c$$

$$3. \frac{40xy^3 - 20xy^2}{-20xy} = -2y^2 + y$$

$$4. \frac{15a^2bc - 12acx^2 + 5ad^2}{-5ac} = \frac{12}{5}x^2 - 3ab - \frac{5d^2}{5ac}$$

$$5. \frac{20ax^3 + 15ax^2 + 10ax + 5a}{5a} = 4x^3 + 3x^2 + 2x + 1$$

According to the rule $\frac{x^3}{x^m} = x^{3-m}$

First let $m=1$; then $\frac{x^3}{x} = x^{3-1} = x^2$

Again let $m=2$; then $\frac{x^3}{x^2} = x^{3-2} = x$

Again let $m=3$; then $\frac{x^3}{x^3} = x^{3-3} = x^0$; but $\frac{x^3}{x^3} = 1$

$\therefore x^0 = 1.$

Again let $m=4$; then $\frac{x^3}{x^4} = x^{3-4} = x^{-1}$; but $\frac{x^3}{x^4} = \frac{1}{x}$

$\therefore \frac{1}{x}$ is the same as x^{-1} ; in like manner it may be shewn that $\frac{1}{x^2}$ is the same as x^{-2} , and that $\frac{1}{x^3}$ is the same as x^{-3} , &c.; and in general that $\frac{1}{x^n}$ is the same as x^{-n} .

In the case of *compound numbers* the objection made to Multiplication does not apply to Division. Thus if 2s. 6d. multiplied by the abstract number 8, give a product of 20s., it follows that 20s. divided by 2s. 6d. must give a quotient = 8. Also, if £2. : 5s. : 4d. be multiplied by 5, the product is £11. : 6s. : 8d.; consequently £11. : 6s. : 8d. divided by £2. : 5s. : 4d. must = 5. Questions of this kind may be solved by various methods; for instance, if it were required to divide £11. : 6s. : 8d. by £2. : 5s. : 4d.

First, £11. : 6s. : 8d. = £11½ = £11.333;

And £2. : 5s. : 4d. = £2. : 5.333s. = £2.2666; and $\frac{11.3333 \text{ \&c.}}{2.2666 \text{ \&c.}} = 5$. Or the question might be more simply solved by reducing both dividend and divisor to the denomination of pence; thus £11. : 6s. : 8d. = 2720d. and £2. : 5s. : 4d. = 544d. and $\frac{2720}{544} = 5$.

If both dividend and divisor be Compound Algebraic Quantities, “arrange the terms of each according to the powers of the same letter. Find how often the first term of the divisor is contained in the first term of the dividend; and place the result in the quotient. Multiply the whole divisor by the quotient thus found: subtract the product from the dividend, and proceed as before till there be no remainder.”





1. Divide
- $a^3 - 3a^2b + 3ab^2 - b^3$
- by
- $a^2 - 2ab + b^2$
- .

$$\begin{array}{r}
 a^3 - 3a^2b + 3ab^2 - b^3 \quad (a - b) \\
 \underline{a^3 - 2a^2b + ab^2} \\
 -a^2b + 2ab^2 - b^3 \\
 \underline{-a^2b + 2ab^2 - b^3} \\
 0
 \end{array}$$

2. Divide
- $a^4 - b^4$
- by
- $a - b$
- .

$$\begin{array}{r}
 a - b \quad a^4 - b^4 \quad (a^3 + a^2b + ab^2 + b^3) \\
 \underline{a^4 - a^3b} \\
 a^3b - b^4 \\
 \underline{a^3b - a^2b^2} \\
 a^2b^2 - b^4 \\
 \underline{a^2b^2 - ab^3} \\
 ab^3 - b^4 \\
 \underline{ab^3 - b^4} \\
 0
 \end{array}$$

3. Divide
- $25x^5 - x^4 - 2x^3 - 8x^2$
- by
- $5x^3 - 4x^2$
- .

$$\begin{array}{r}
 5x^3 - 4x^2 \quad 25x^5 - x^4 - 2x^3 - 8x^2 \quad (5x^2 + 4x^3 + 3x + 2) \\
 \underline{25x^5 - 20x^4} \\
 20x^4 - x^4 - 2x^3 - 8x^2 \\
 \underline{20x^4 - 16x^3} \\
 15x^3 - 2x^3 - 8x^2 \\
 \underline{15x^3 - 12x^2} \\
 10x^2 - 8x^2 \\
 \underline{10x^2 - 8x^2} \\
 0
 \end{array}$$

4. Divide $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ by $x^2 - 2x + 1$.
5. Divide $x^3 - nax^2 + na^2x - a^3$ by $x - a$.
6. Divide 1 by $1 - x$.
7. Divide a^3 by $1 - x^3$.
8. Divide $a^n - b^n$ by $a - b$.
9. Divide $1 + x + x^2 + x^3 + \&c.$ by $1 - x$.
10. Divide 1 by $1 - 2x + x^2$.
11. Divide $x^3 - px^2 + qx - r$ by $x - b$.
12. Divide $a^6 + b^6$ by $a + b$.
13. Divide $x^4 - y^4$ by $x - y$.
14. Divide $x^6 - y^6$ by $x + y$.
15. Divide $x^{2n} - y^{2n}$ by $x + y$.
16. Divide $x^{2n+1} + y^{2n+1}$ by $x + y$.

Ans. to 4. $x^3 - 3x^2 + 3x - 1$

Ans. to 5. $x^2 - \overline{n-1}.ax + a^2 \quad \frac{x^n}{1-x}$

Ans. to 6. $1 + x + x^2 + x^3 + \dots \quad \frac{x^n}{1-x}$

Ans. to 7. $a^3 \left(1 + \cancel{x^3} x^6 + \&c. \quad \frac{x^n}{1-x^3} \right)$

Ans. to 8. $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \&c. \quad \frac{a^n - b^n}{a - b}$

Ans. to 9. $1 + 2x + 3x^2 + 4x^3 + \dots \quad \frac{nx}{1-x}$

Ans. to 10. $1 - 1 + 1 - 1 + 1 - \&c. \quad \frac{1}{1-x^2}$

Ans. to 11. $x - \overline{p-b}.x + q + \overline{pb-b^2}. \quad \frac{q+pb-b^2}{x-b}$

Ans. to 12. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$

Ans. to 13. $x^4 + x^3y + x^2y^2 + y^3 + \overline{y^4 - y^6} \quad \frac{x-y}{x^2-y^2}$

Ans. to 14. $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$

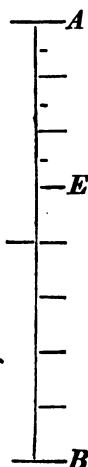
Ans. to 15. $x^{2n-1} + x^{2n-2}y + x^{2n-3}y^2 + \dots + y^{2n}$

Ans. to 16. $x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \dots + y^{2n+1}$



FRACTIONS.

By the fraction $\frac{a}{b}$, I mean that unity is divided into (b) parts, and that (a) of those parts are taken. Thus, if the line AB represent a yard, and it be divided into 8 equal parts, and AE be 3 of those parts, AE is $\frac{3}{8}$ of AB . Again, divide this line AB into 16 equal parts, by which means we bisect each of the former parts, and the same line AE is now become $\frac{6}{16}$ of AB . Hence $\frac{6}{16}$ must be the same as $\frac{3}{8}$, i.e. the value of the fraction remains unaltered, when both its numerator and denominator are multiplied or divided by the same quantity.



Addition of fractional quantities, which have a common denominator, is performed by adding together the numerators, and subscribing the common denominator. Thus, if as before, AB be one yard, and AE be $\frac{3}{8}$ of AB , the remaining line EB will be $\frac{5}{8}$ of AB ; and $\frac{3}{8} + \frac{5}{8} = \frac{8}{8} = 1$ yard, the whole line AB .

If the fractions have not a common denominator, they must be transformed into equivalent ones that have. Thus, to add together $\frac{1}{2}$ and $\frac{1}{3}$, we transform these fractions into others which have 6 for a common denominator; thus $\frac{2}{3} + \frac{1}{3} = \frac{3}{3}$, where $\frac{2}{3}$ is, as has been shewn, the same as $\frac{1}{2}$.

Add together $\frac{1}{2}$, $\frac{3}{10}$ and $\frac{1}{10}$; these fractions are the same as $\frac{4}{20}$, $\frac{6}{20}$, $\frac{2}{20}$, and their sum is $\frac{12}{20}$. The common rule for reducing different fractions to others which shall have a

common denominator is, to “multiply the numerator of each fraction into all the denominators except its own, for a new denominator; and all the denominators together for a new denominator.” Thus, if I wish to add together $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$; First,

$$\left. \begin{array}{l} 1 \times 3 \times 4 = 12 \\ 1 \times 2 \times 4 = 8 \\ 1 \times 2 \times 3 = 6 \end{array} \right\} \text{New numerators.}$$

$$2 \times 3 \times 4 = 24, \quad \text{Common denominator.}$$

The proposed fractions then are reduced to these equivalent ones, $\frac{12}{24}$, $\frac{8}{24}$, $\frac{6}{24}$; for $\frac{12}{24}$ is the same as $\frac{1}{2}$; $\frac{8}{24}$ is the same as $\frac{1}{3}$; $\frac{6}{24}$ is the same as $\frac{1}{4}$; we have now, as before, to add the numerators together, and subscribe the common denominator, thus

$$\frac{12 + 8 + 6}{24} = \frac{26}{24} = \frac{13}{12}$$

Add together $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{5}{12}$;

$$\left. \begin{array}{l} 2 \times 2 \times 4 = 16 \\ 1 \times 3 \times 4 = 12 \\ 3 \times 2 \times 3 = 18 \end{array} \right\} \text{New numerators.}$$

$$3 \times 2 \times 4 = 24 \quad \text{Common denominator.}$$

$$\frac{16 + 12 + 18}{24} = \frac{46}{24} = \frac{23}{12}$$

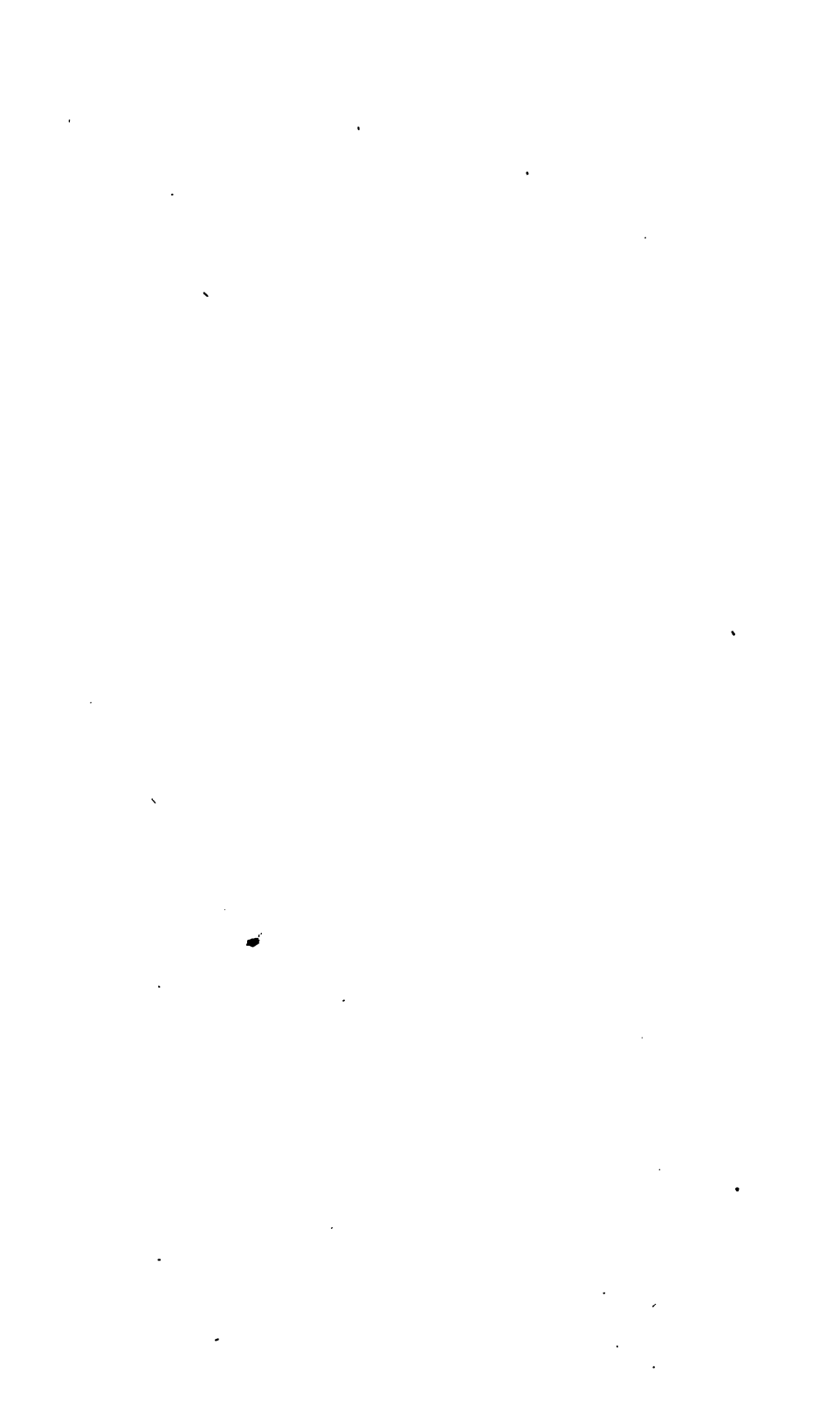
Add together $\frac{2x}{5}$ and $\frac{3x}{7}$

$$\left. \begin{array}{l} 2x \times 7 = 14x \\ 3x \times 5 = 15x \end{array} \right\} \text{New numerators;}$$

$$5 \times 7 = 35 \quad \text{Common denominator.}$$

$$\frac{14x + 15x}{35} = \frac{29x}{35}$$









Add together $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$;

$$\left. \begin{array}{l} x \times 3 \times 4 = 12x \\ x \times 2 \times 4 = 8x \\ x \times 2 \times 3 = 6x \end{array} \right\} \text{ new numerators.}$$

$$2 \times 3 \times 4 = 24 \quad \text{common denominator.}$$

$$\frac{12x + 8x + 6x}{24} = \frac{26x}{24} = \frac{13x}{12}$$

Add together $\frac{4x}{7}$ and $\frac{x-2}{5}$.

$$\left. \begin{array}{l} 4x \times 5 = 20x \\ x-2 \times 7 = 7x-14 \end{array} \right\} \text{ new numerators.}$$

$$7 \times 5 = 35 \quad \text{common denominator.}$$

$$\frac{20x + 7x - 14}{35} = \frac{27x - 14}{35}.$$

If a fraction is to be added to a whole number, we may either consider the whole number as a fraction whose denominator is unity; or multiply the whole number into the denominator of the fraction, and to this product add the numerator. Thus $3 + \frac{2}{5}$ may be considered as $\frac{3}{1} + \frac{2}{5}$.

$$\left. \begin{array}{l} 3 \times 5 = 15 \\ 2 \times 1 = 2 \end{array} \right\} \text{ new numerators.}$$

$$1 \times 5 = 5 \quad \text{common denominator.}$$

$$\frac{2 + 15}{5} = \frac{17}{5}.$$

Or thus $\frac{3 \times 5 + 2}{5} = \frac{17}{5}.$

$$2a + \frac{3x}{5} = \frac{10a + 3x}{5}$$

$$x - \frac{4y}{7} = \frac{7x - 4y}{7}$$

Add together $5x + \frac{x-2}{3}$ and $4x - \frac{2x-3}{5x}$.

We may here employ the same method; or add together the whole numbers separately, and the fractions separately.

Thus $\frac{15x+x-2}{3} + \frac{20x^2-2x-3}{5x}$; these fractions, when reduced, become $\frac{75x^3+5x^2-10x}{15x} + \frac{60x^2-6x-9}{15x} =$
 $\frac{140x^2-16x+9}{15x} = \frac{135x^2}{15x} + \frac{5x^2-16x+9}{15x} =$
 $9x + \frac{5x^2-16x+9}{15x}$. Or more simply thus $5x + 4x = 9x$
 $\frac{x-3}{3} - \frac{2x-3}{5x} = \frac{5x^2-15x}{15x} - \frac{6x-9}{15x} = \frac{5x^2-16x+9}{15x}$
 $\therefore$ sum of the fractions is $9x + \frac{5x^2-16x+9}{15x}$, as before.

Add together $2a$, $3a + \frac{2x}{5}$ and $a - \frac{8x}{9}$.

First $2a + 3a + a = 6a$,

Then $\frac{2x}{5} - \frac{8x}{9} = \frac{18x-40x}{45} = -\frac{22x}{45}$

$\therefore$ the sum required is $6a - \frac{22x}{45} = \frac{270a-22x}{45}$.

The Rule for Subtraction of Fractions is the same as for Addition, except that we take the *difference*, instead of the sum, of the numerators.

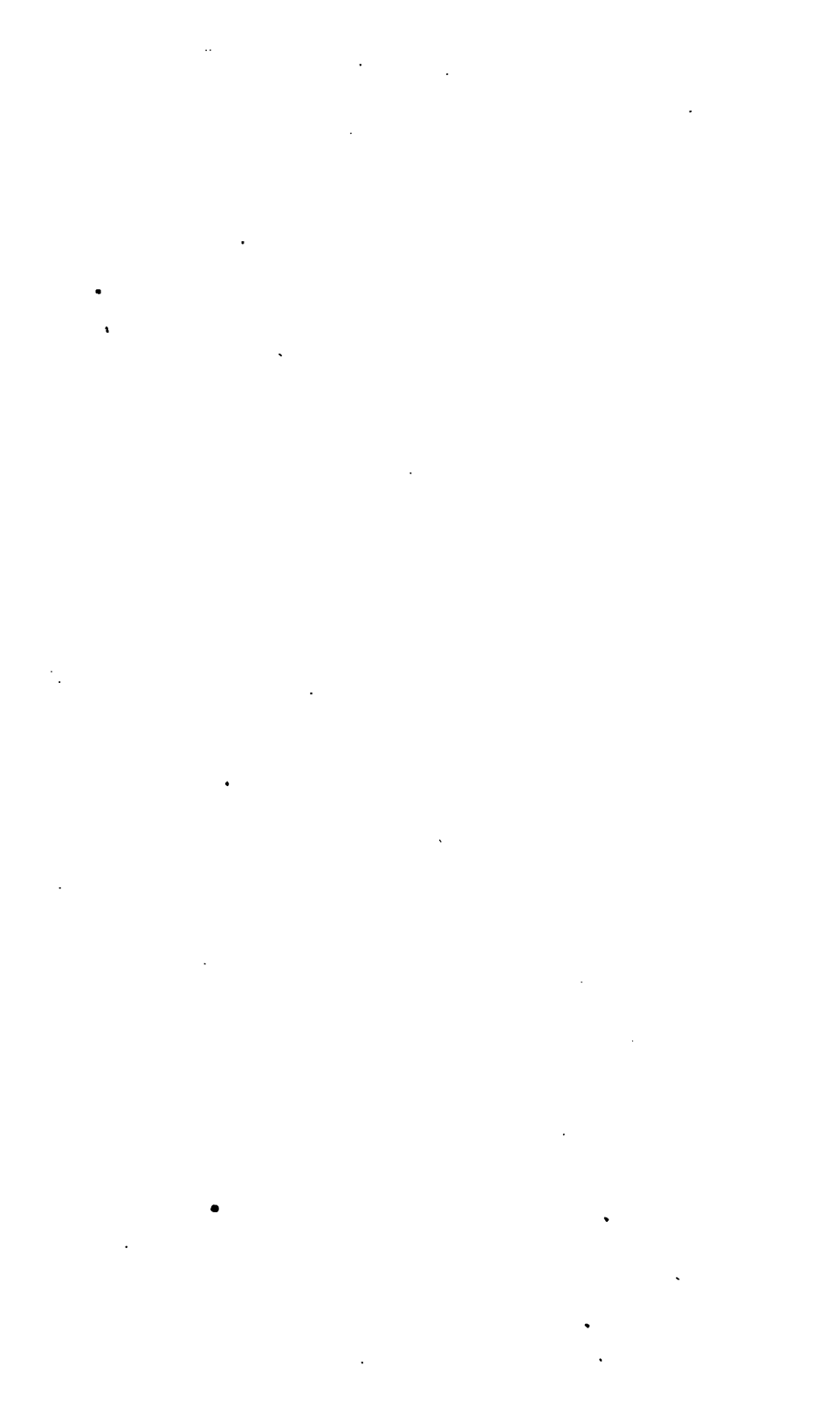
Thus, take $\frac{12a}{x}$ from $\frac{6ax}{5}$.

$$\left. \begin{array}{l} 12a \times 5 = 60a \\ 6ax \times x = 6ax^2 \end{array} \right\} \text{new numerators.}$$

$x \times 5 = 5x$ common denominator.

$$\frac{6ax^2}{5x} - \frac{60a}{5x} = \frac{6ax^2-60a}{5x}.$$





From $\frac{9x}{2}$ take $\frac{2x+1}{3}$.

$$9x \times 3 = 27x$$

$$2 \times (2x+1) = 4x+4$$

$$3 \times 2 = 6$$

$$\frac{27x}{6} - \frac{4x+4}{6} = \frac{27x-4x-4}{6} = \frac{23x-4}{6}$$

From $\frac{1}{x-y}$ take $\frac{1}{x+y}$.

$$1 \times \frac{x+y}{x+y} = x+y$$

$$1 \times \frac{x-y}{x-y} = x-y$$

$$x-y \times x+y = x^2-y^2$$

$$\frac{x+y}{x^2-y^2} - \frac{x-y}{x^2-y^2} = \frac{x-x+y+y}{x^2-y^2} = \frac{2y}{x^2-y^2}$$

EXAMPLES

For practice in Addition and Subtraction of Algebraic Fractions.

1. Add together $\frac{2b}{x^2+b^2}$ and $\frac{1}{x}$.

$$\frac{x+b}{x} \cdot \frac{x^2+b^2}{x^2+b^2}$$

2. Add together $\frac{x+y}{x-b}$ and $\frac{x-1}{x+y}$.

$$\frac{(2y+b+1) \cdot x + x \cdot x - b + y}{x \cdot x - b + y}$$

3. Add together $5x$, $\frac{2a}{3x^2}$ and $\frac{a+2x}{4x}$.

$$\frac{x \cdot (60x^2 + 3a + 6)}{12x^2}$$

4. Add together $\frac{3x}{7}$, $\frac{5x}{9}$ and $\frac{4x}{11}$.

$$\frac{934x}{693} = \frac{37a^2 + 11b}{156}$$

5. Add together $\frac{5a^2+b}{3b}$ and $\frac{4a^2+2b}{5b}$.

6. Add together $\frac{3x+2}{a}$, $\frac{4x+3}{b}$ and $\frac{5x+4}{c}$.

$$\frac{x \cdot 136c + 4ac}{abc}$$

7. Add together $\frac{a}{x}$, $\frac{ax}{x^2-a^2}$ and $\frac{a^2}{x^2+a^2}$.

$$\frac{a(2x^4 - a^4 + a^3x^2 + a^2)}{x(x^4 - a^4)}$$

8. From $\frac{6ax}{5}$ subtract $\frac{12x}{a}$. $\frac{(6a^2 + 60)x}{5a^2}$

9. From $\frac{5x-3}{x+1}$ subtract $\frac{3x+2}{x-1}$. $\frac{2x^2 - 13x + 1}{x^2 - 1}$

10. Find the difference of $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. $\frac{4xy}{x^2 - y^2}$

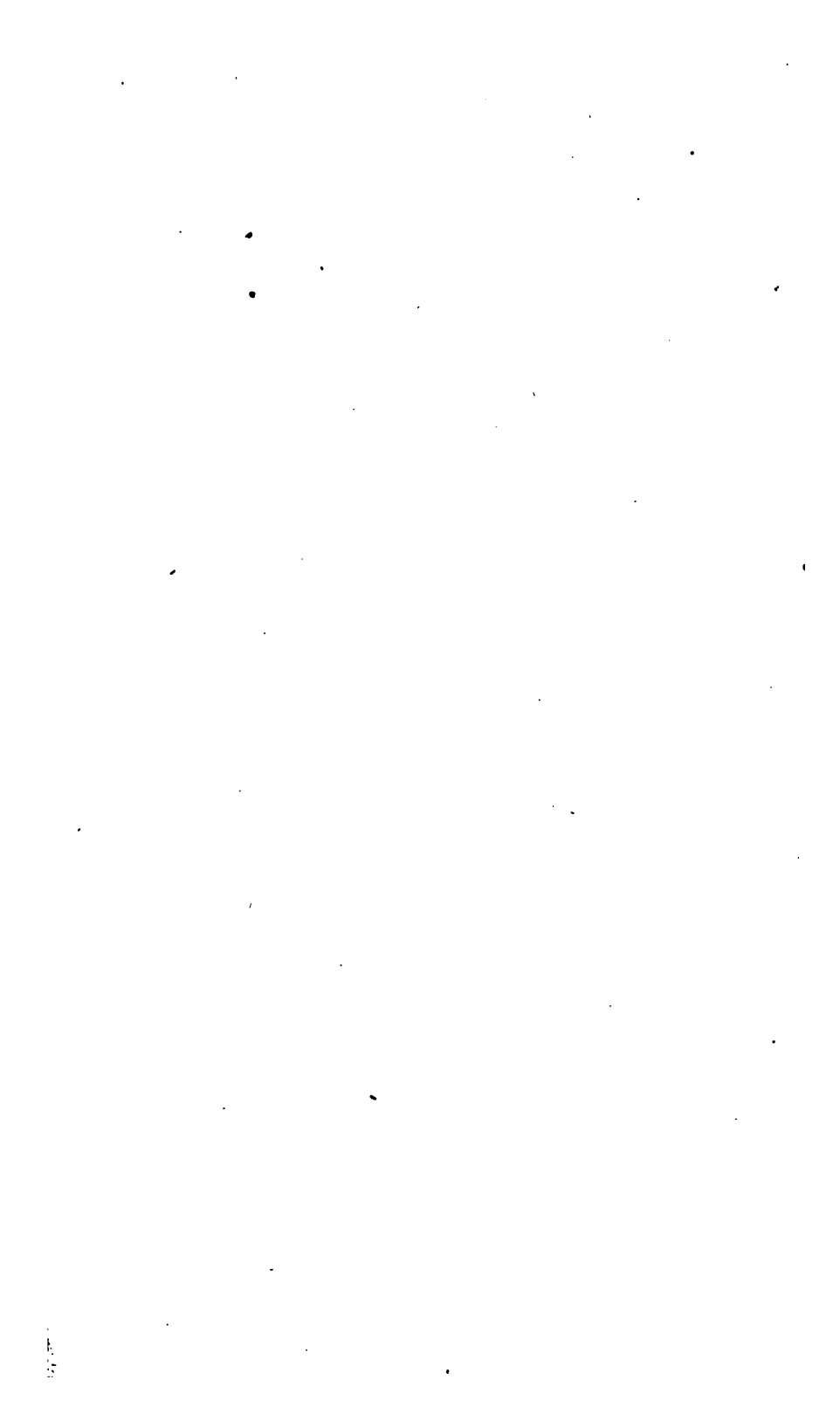
11. From $x - \frac{x^2}{2a+x}$ subtract $a - \frac{a^2}{2x+a}$. $\frac{2(a^2 - x^2)}{5ax + 2a^2 + 2x}$

12. From $a + \frac{x-a}{2b}$ subtract $2a - \frac{2a-4x}{3c}$. $\frac{3cx - 4ab - 3ac}{6bc}$

13. From $ax + \frac{a+x}{a(a-x)}$ subtract $a + \frac{a-x}{a(a+x)}$.

Ans. $\frac{a^2x - ax^3 - a^3 + ax^2 + 4x}{a^2 - x^2}$

or $a(x - a) + \frac{4x}{a^2 - x^2}$





MULTIPLICATION

AND

DIVISION OF FRACTIONS.

A FRACTION is multiplied by a whole number, either by multiplying the numerator, or dividing the denominator of the Fraction by the whole number. Thus, $\frac{4}{9} \times 3$ is $\frac{4}{9}$ added three times to itself, as $\frac{4+4+4}{9} = \frac{12}{9} = \frac{4}{3}$, the same result as if we had divided the denominator of the Fraction by 3. Also $\frac{7}{48} \times 8 = \frac{7}{6}$, the same result as $\frac{7}{48 \div 8}$.

A Fraction is divided by a whole number, either by dividing the numerator, or multiplying the denominator by the whole number. Thus, $\frac{12}{37} \div 3 = \frac{4}{37}$, or $\frac{12}{37} (= \frac{4}{37})$; for $\frac{4}{37} \times 3 = \frac{12}{37}$, as has been just shewn.

One Fraction is multiplied by another by multiplying the numerators together for a new numerator, and the denominators together for a new denominator. Thus, $\frac{3}{4} \times \frac{5}{6} = \frac{15}{24} = \frac{5}{8}$. For $\frac{3}{4} \times 5 = \frac{15}{4}$; but since the multiplier (5) is here 6 times as great as it ought to be, in order to obtain the true product, we must divide the result $\frac{15}{4}$ by 6; i. e. the product sought is $\frac{15}{4} \div 6 = \frac{5}{8}$.

To divide one fraction by another, invert the divisor, and then proceed as in Multiplication. Thus $\frac{7}{6} \div \frac{4}{3} = \frac{7}{6} \times \frac{3}{4} = \frac{21}{24}$; for we know that $\frac{21}{24} \times \frac{4}{3} = \frac{28}{12} = \frac{7}{3}$.

Multiply $\frac{3x}{2}$ by $\frac{5x}{3b}$.

$$\frac{3x}{2} \times \frac{5x}{3b} = \frac{15x^2}{6b} = \frac{5x^2}{2b}.$$

Multiply $\frac{1}{2}$ of 7 by $\frac{1}{4}$ of x .

$$\frac{7}{2} \times \frac{x}{4} = \frac{7x}{4}.$$

Multiply $\frac{2x}{3}$ by $\frac{3x}{4}$.

$$\frac{2x}{3} \times \frac{3x}{4} = \frac{2 \times 3 \times x^2}{3 \times 4} = \frac{x^2}{2}.$$

Multiply $\frac{2}{9}$ of $\frac{3x}{4}$ by $\frac{5}{8}$ of $\frac{3x}{7}$.

$$\frac{2}{9} \text{ of } \frac{3x}{4} = \frac{x}{6}; \quad \frac{5}{8} \text{ of } \frac{3x}{7} = \frac{15x}{56};$$

$$\frac{x}{6} \times \frac{15x}{56} = \frac{15x^2}{6 \times 56} = \frac{5x^2}{2 \times 56} = \frac{5x^2}{112}.$$

Multiply $\frac{a^2-x^2}{a-b}$ and $\frac{a^2-b^2}{a-x}$ together.

$$\frac{a^2-x^2}{a-b} \times \frac{a^2-b^2}{a-x} = \frac{\overline{a+x} \cdot \overline{a-x} \cdot \overline{a+b} \cdot \overline{a-b}}{\overline{a-b} \cdot \overline{a-x}} = \overline{a+x} \cdot \overline{a+b}$$



EXAMPLES

For practice in Multiplication of Algebraic Fractions.

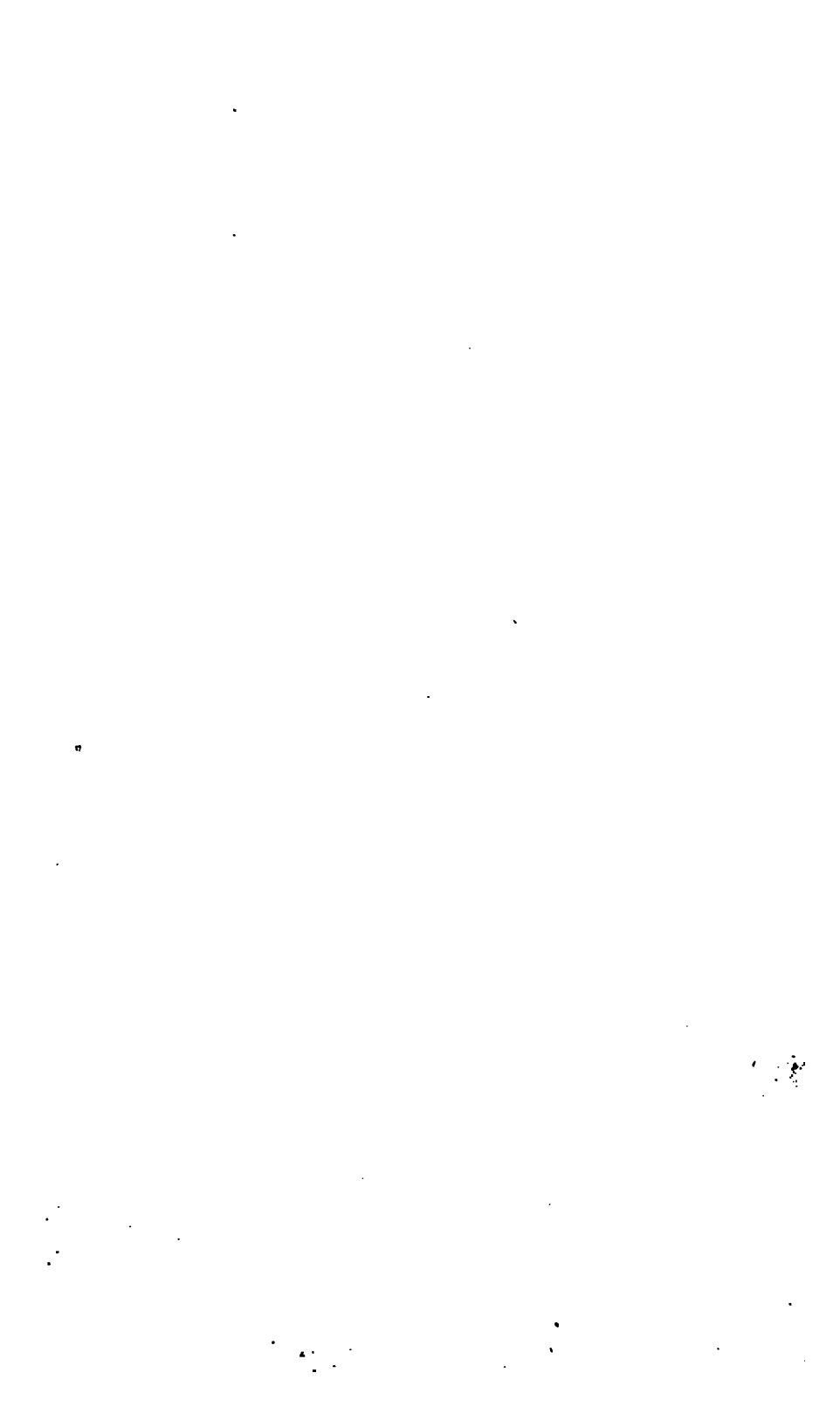
1. Multiply $\frac{a^3-x^3}{a^3+x^3}$ by $\frac{\overline{a+x}^2}{\overline{a-x}^2} \cdot \frac{a^2+ax+x^2}{a^2-ax+x^2}$

2. Multiply $\frac{a+x}{a}$, $\frac{a-x}{x}$ and $\frac{a^2-x^2}{a^2+x^2}$ together. $\frac{a^2-2a^2x^2}{a^2+x^2}$

3. Multiply $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$. $\frac{9x}{10x-20}$

Multiply $3x$, $\frac{x+1}{2a}$ and $\frac{x-1}{a+b}$ together. $\frac{3x^3-3x}{2a^2+2a}$





5. Multiply $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$ and $a+\frac{ax}{a-x}$ together. $\frac{a^2/a}{x^2}$

6. Multiply $\left(\frac{a-x}{a+x} - \frac{ax}{a^2-x^2}\right)$ by $\frac{a-x}{a-3x} \cdot \frac{a+3x}{a-3x} \cdot \frac{(a-x)^2-aa}{(a+x)(a-x)}$

7. Multiply $\frac{(a+b)^m}{x^2-y^2}$ by $\frac{x^2-y^2}{a+b} \cdot \frac{a+b}{x^2-y^2}$

EXAMPLES

For practice in Division of Algebraic Fractions.

1. Divide $\frac{7}{4x}$ by $4x$.

$$\frac{7}{4x} \div 4x = \frac{7}{32x}$$

2. Divide $\frac{x+a}{x-b}$ by $\frac{x+a}{3x-3b}$.

$$\frac{x+a}{x-b} \div \frac{x+a}{3x-3b} = \frac{x+a}{x-b} \times \frac{3x-3b}{x+a} = \frac{x+a \times 3 \times x-b}{x+a \cdot x-b} = 3.$$

3. Divide $\frac{a}{1-a}$ by $\frac{a}{5}$.

$$\frac{a}{1-a} \div \frac{a}{5} = \frac{a}{1-a} \times \frac{5}{a} = \frac{5}{1-a}.$$

4. Divide $\frac{x-a}{x+b}$ by $\frac{x^2-a^2}{x+b}$.

$$\begin{aligned} \frac{x-a}{x+b} \div \frac{x^2-a^2}{x+b} &= \frac{x-a}{x+b} \times \frac{x+b}{x^2-a^2} = \frac{x-a \times x+b \cdot x+b}{x-a \cdot x+a \cdot x+b} \\ &= \frac{x+b}{x+a}. \end{aligned}$$

5. Divide $\frac{2ax^2+x^3}{a^3-x^3}$ by $\frac{x^4}{a-x}$.

$$\frac{2ax^2+x^3}{a^3-x^3} \div \frac{x^4}{a-x} = \frac{2ax^2+x^3}{a^3-x^3} \times \frac{a-x}{x^4} = \frac{\overline{2a+x} \cdot x^3}{a-x \times (a^2+ax+x^2)} \times \frac{a-x}{x^4} = \frac{2a+x}{x^3 \cdot (a^2+ax+x^2)}.$$

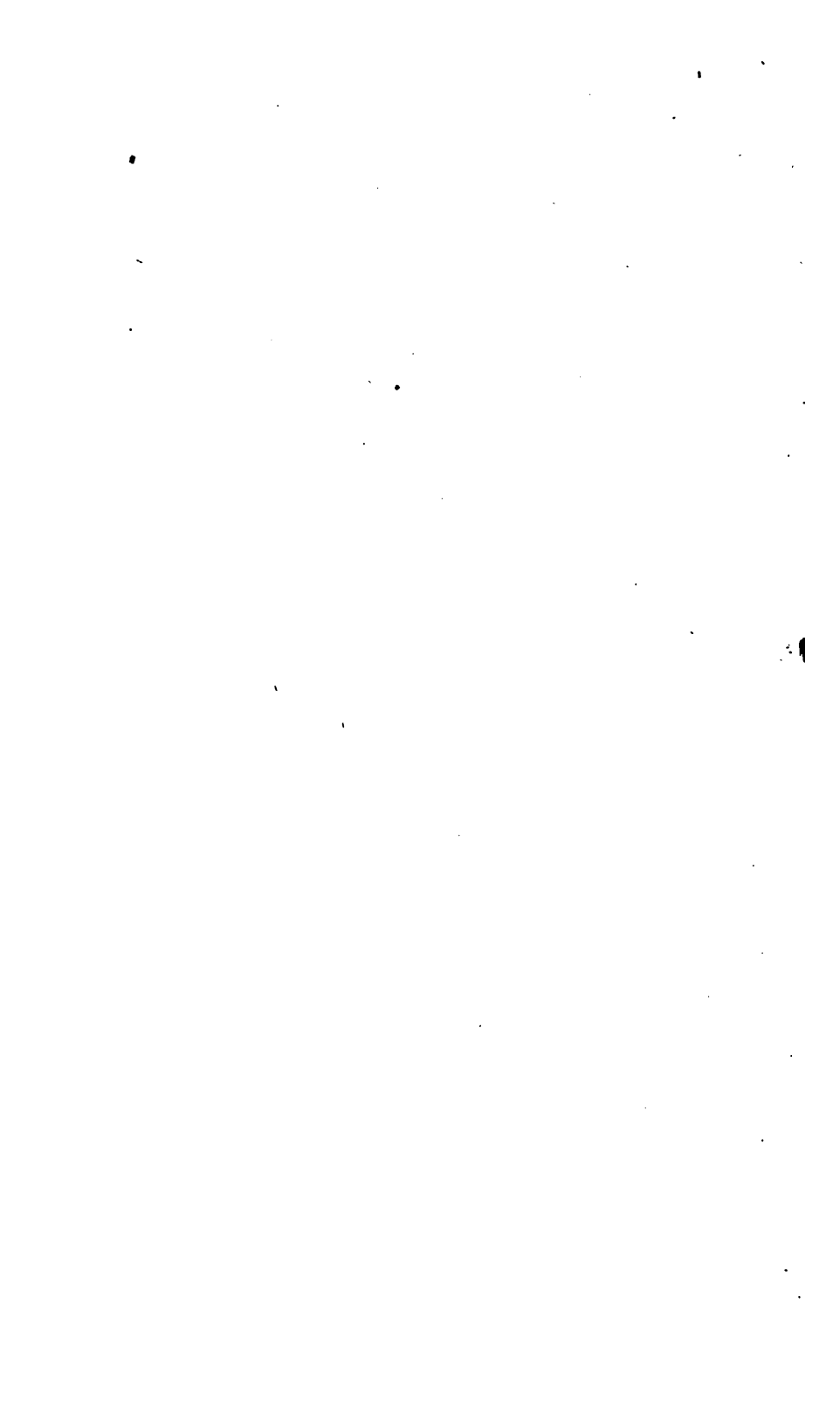
6. Divide $\frac{a^4-b^4}{(x+y)^3}$ by $\frac{a^2+b^2}{x^2+2xy+y^2}$.

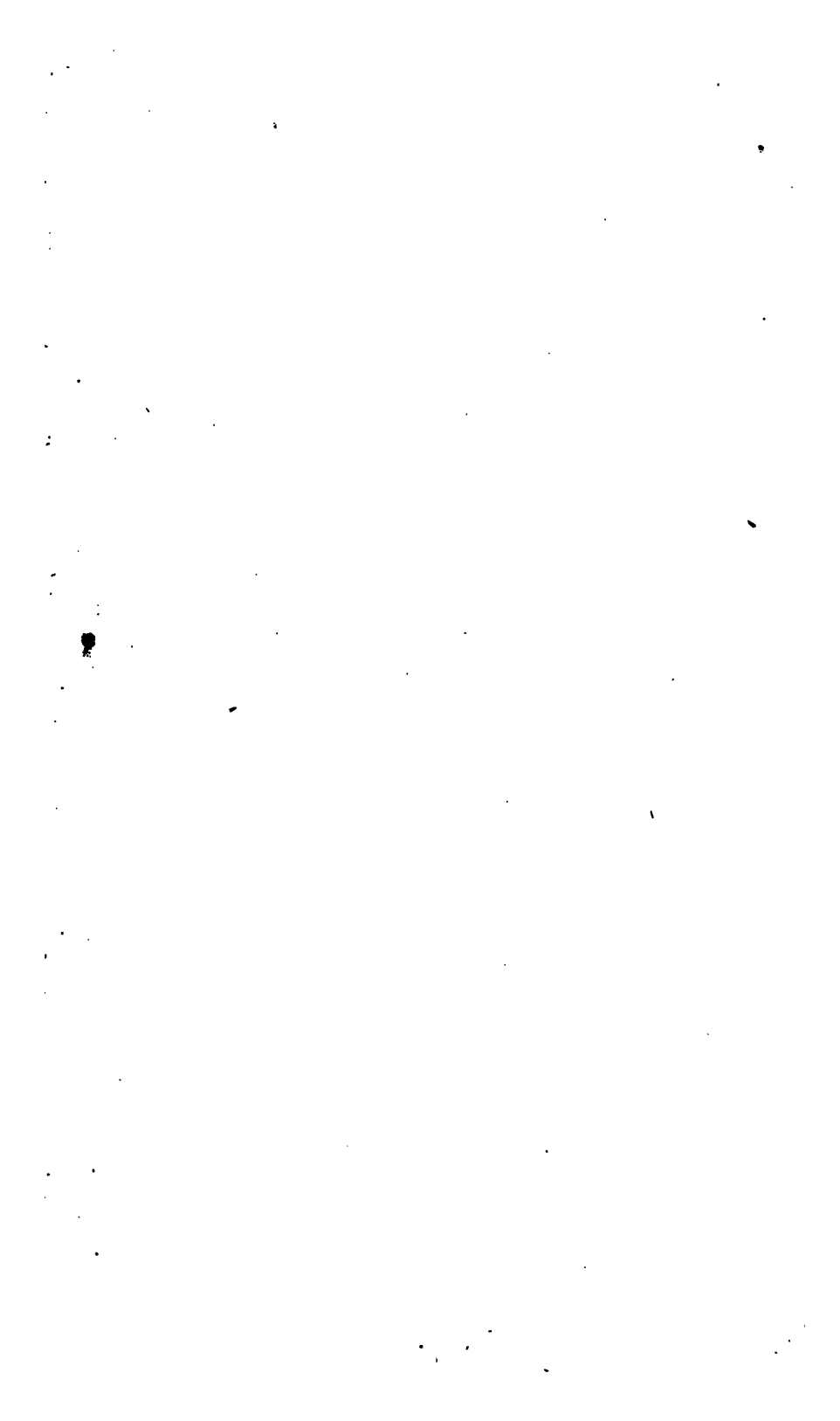
$$\frac{a^4-b^4}{(x+y)^3} \div \frac{a^2+b^2}{x^2+2xy+y^2} = \frac{a^4-b^4}{(x+y)^3} \times \frac{(x+y)^2}{a^2+b^2} = \frac{(a^2+b^2) \cdot (a^2-b^2)}{(x+y)^3} \times \frac{(x+y)^2}{a^2+b^2} = \frac{a^2-b^2}{x+y}.$$

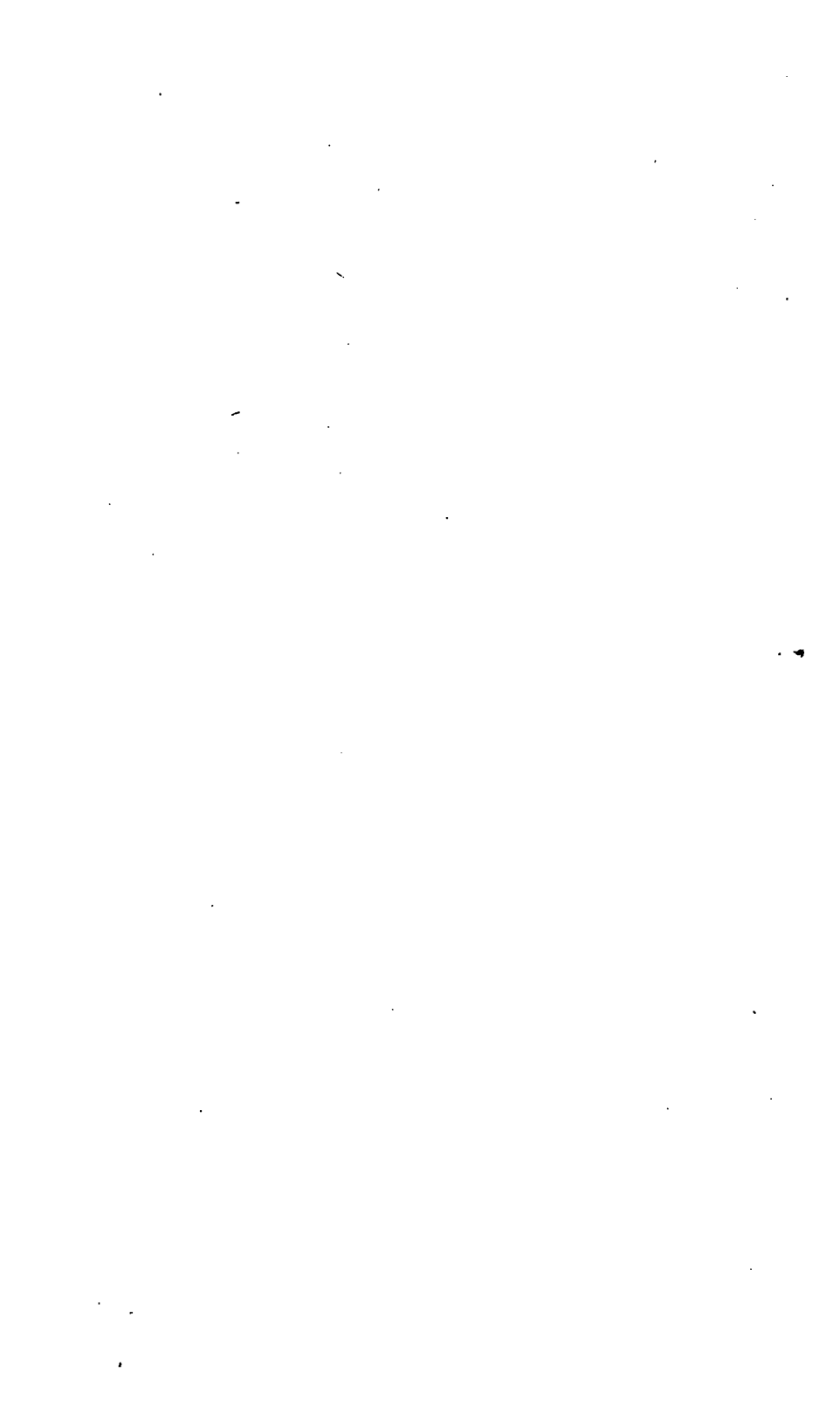
7. Divide $\frac{\overline{a-x}^{\frac{7}{3}}}{(x+y)^{\frac{3}{4}}}$ by $\frac{\overline{a-x}^{\frac{10}{7}}}{(x+y)^{\frac{1}{5}}}$. $\frac{x+y}{a-x}$

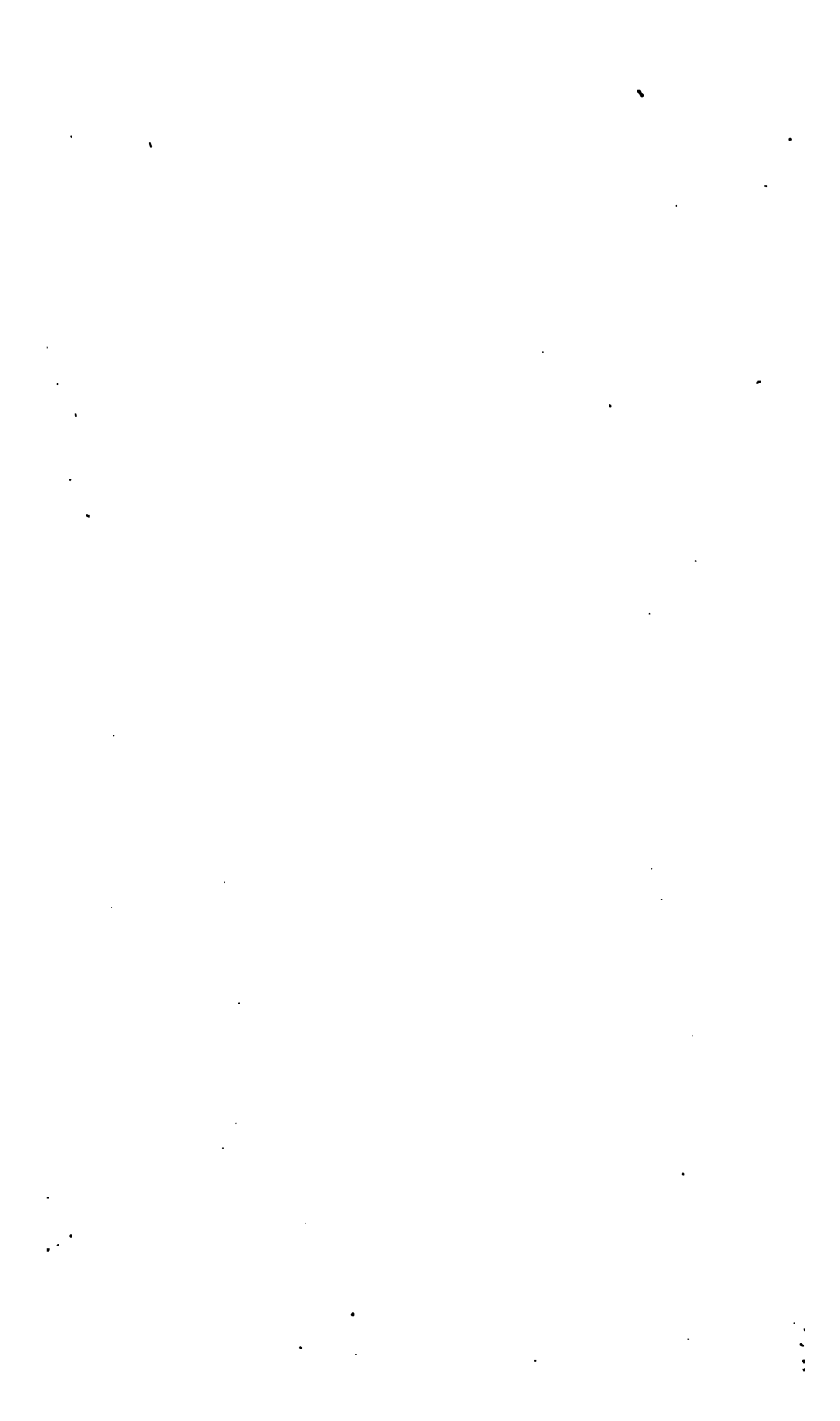
8. Divide $\frac{\overline{a+b}^{\frac{1}{4}}}{(x-y)^{\frac{1}{4}}}$ by $\frac{\overline{a+b}^3}{(x-y)^3}$. $\frac{\overline{x-y}^{\frac{11}{4}}}{a+b} \cdot \frac{1}{x-y}$

9. Divide $\frac{\sqrt[2]{a^3+b^3}}{\sqrt[5]{x+y}}$ by $\frac{\sqrt[3]{a^3+b^3}}{\sqrt[5]{x+y}}$. $\frac{\sqrt[4]{x+y}}{a^2-b^2} \cdot \frac{a-3}{2}$









RATIO AND PROPORTION.

Definition.

RATIO is the relation, in point of magnitude, which one quantity bears to another of the same kind. For instance, the ratio of (*a*) to (*b*), expressed thus (*a* : *b*), denotes the magnitude of (*a*) with respect to (*b*). And when two ratios are equal to each other, the four quantities composing them are said to be *proportionals*. Thus if the ratio of *a* : *b* = the ratio of *c* : *d*, the quantities *a*, *b*, *c*, *d*, are said to constitute a *proportion*, and they are written thus, *a* : *b* :: *c* : *d* where (*a*) and (*c*) are called the *antecedents*, and (*b*) and (*d*) the *consequents*. It appears from the definition that if *a*, *b*, *c*, *d* be proportionals, $\frac{a}{b} = \frac{c}{d}$.

Hence we deduce the following conclusions.

$$\text{Since } 1 = 1 \quad (M)$$

$$\text{And } \frac{a}{b} = \frac{c}{d} \quad (N)$$

$$\therefore \frac{b}{a} = \frac{d}{c} \quad \left(\frac{M}{N}\right)$$

$$\therefore b : a :: d : c.$$

$$\text{Again, since } \frac{a}{b} = \frac{c}{d}$$

$$\text{And } \frac{b}{c} = \frac{b}{c}$$

$$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$$

$$\text{Or } \frac{a}{c} = \frac{b}{d} \therefore a : c :: b : d.$$

Again, since $\frac{a}{b} = \frac{c}{d}$

$$1 = 1$$

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$\text{Or } \frac{a+b}{b} = \frac{c+d}{d} \quad \therefore a+b : b :: c+d : d.$$

Again, since $\frac{a}{b} = \frac{c}{d}$

$$1 = 1$$

$$\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$$

$$\frac{a-b}{b} = \frac{c-d}{d} \quad \therefore a-b : b :: c-d : d.$$

Again, since $\frac{a-b}{b} = \frac{c-d}{d}$ (M)

$$\text{And } \frac{a}{b} = \frac{c}{d} \quad (N)$$

$$\therefore \frac{a-b}{a} = \frac{c-d}{c} \quad \left(\frac{M}{N}\right) \quad \therefore a-b : a :: c-d : c.$$

Again, since $\frac{a-b}{b} = \frac{c-d}{d}$ (A)

$$\text{And } \frac{a+b}{b} = \frac{c+d}{d} \quad (B)$$

$$\therefore \frac{a-b}{a+b} = \frac{c-d}{c+d} \quad \left(\frac{A}{B}\right)$$

$$\therefore a-b : a+b :: c-d : c+d.$$

If $a : b :: l : m$

And $b : c :: n : p$

And $c : d :: q : r$

Then $a : d :: l \times n \times q : m \times p \times r$





$$\text{For } \frac{a}{b} = \frac{l}{m}$$

$$\text{And } \frac{b}{c} = \frac{n}{p}$$

$$\text{And } \frac{c}{d} = \frac{q}{r}$$

$$\therefore \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{l}{m} \times \frac{n}{p} \times \frac{q}{r}$$

$$\text{Or } \frac{a}{d} = \frac{lnq}{mpr} \therefore a : d :: lnq : mpr.$$

On this principle of compounding proportions is founded the rule for solving Arithmetical Questions in *Double Rule of Three*.

Since when a, b, c, d are proportionals,

$$\frac{a}{b} = \frac{c}{d} \therefore d = \frac{bc}{a}.$$

Upon this proof is founded the rule for solving Arithmetical Questions in *Single Rule of Three*.

Since then *Proportion* consists of two equal ratios, and that the two terms of any ratio must be of the same denomination, we will proceed to apply this principle to a few cases of *Rule of Three*.

1. If 14 lbs. of sugar cost 10s., what is the price of 2 lbs.?

Let x = price sought, and place it the last term of the proportion. The *given* quantity of the same denomination with (x) will of course be the third term. Then consider, from the nature of the question, whether the third term be greater or less than the fourth, and as the first will accordingly be greater or less than the second, this determines the order of the two other terms. In the case before us, we have

14 lbs. cost 10s.

2 lbs. cost x .

And since it is manifest that x is less than 10, and as x is always the last term, 2 must be the second term of the first ratio.

Thus $14 : 2 :: 10 : x$

Or $7 : 1 :: 10 : x \therefore 7x = 10 \therefore x = \frac{10}{7} = 1 : 5\frac{1}{7}$.

2. If my board cost a guinea a week; how many months board can I have for £100?

Let x = time sought.

21s. 1 week.

2000s. x week.

Here it is evident that x is greater than one week, therefore 2000 must be the second term of the proportion,

$21 : 2000 :: 1 : x$

$\therefore 21x = 2000 \therefore x = \frac{2000}{21} = 95\frac{5}{21} = 23 : 3 : 1\frac{2}{3}$

3. What length of cloth $\frac{3}{4}$ yard wide will cover a floor $3\frac{3}{4}$ yards long, and $1\frac{1}{2}$ yard wide?

Let x = length required.

Length.	Width.
$3\frac{3}{4}$	$1\frac{1}{2}$
x	$\frac{3}{4}$

The narrower the cloth the greater must be its length to cover a given surface; therefore x is greater than $3\frac{3}{4}$.

$\frac{3}{4} : 1\frac{1}{2} :: 3\frac{3}{4} : x$

Or $\frac{3}{4} : \frac{3}{2} :: \frac{15}{4} : x$

$\therefore 3 : \frac{3}{2} :: 15 : x$

$\therefore 3 : 3 :: 15 : 2x \therefore 2x = 15 \therefore x = 7\frac{1}{2}$.

4. If a family of 9 persons spend £120 in 8 months; how much money will serve 24 persons for 16 months, at the same rate?

Let x = the money sought.

Persons.	£.	Months.
9	120	8
24	x	16

$9 : 24 :: 120 : y$, money spent by 24 persons in 8 mths.

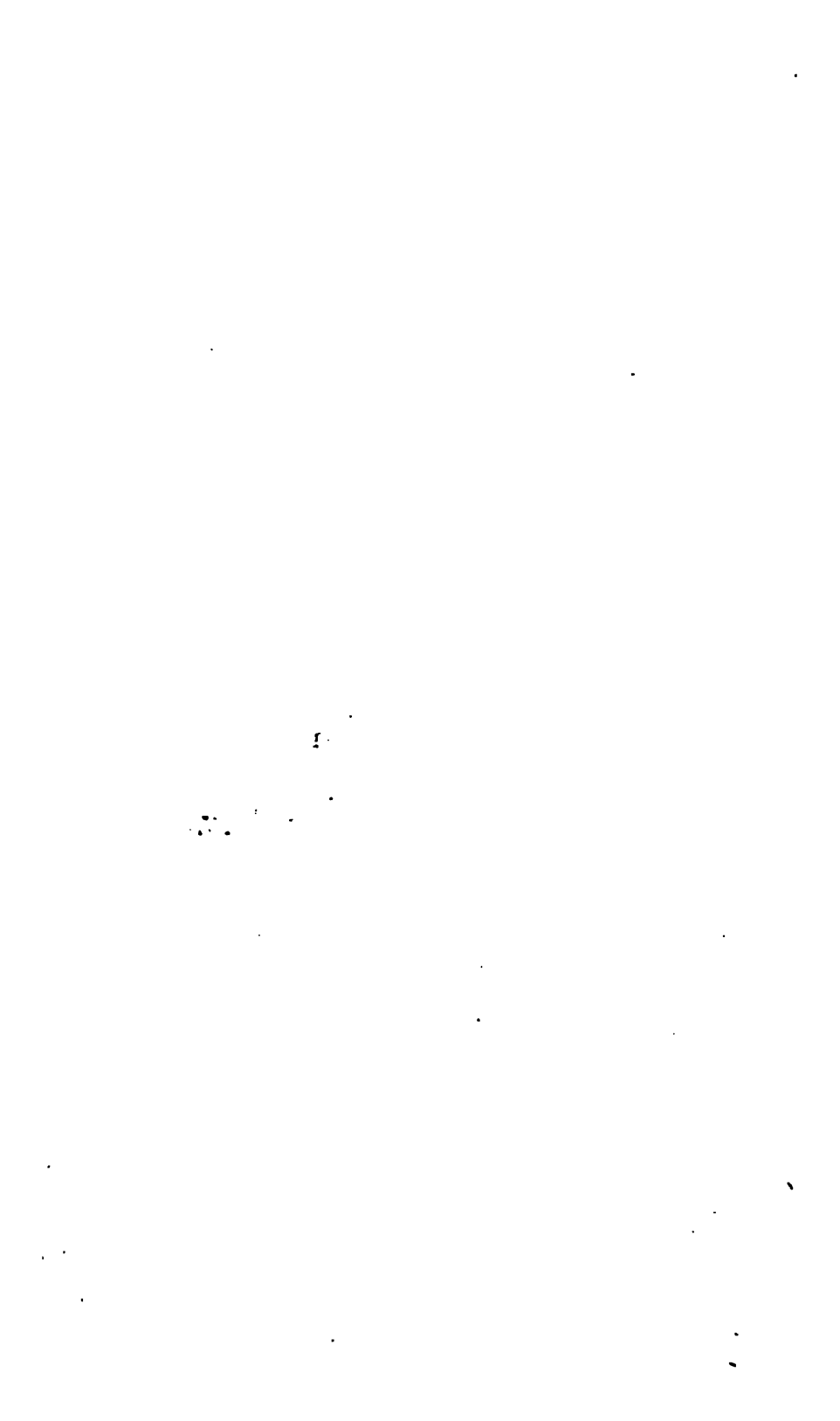
$8 : 16 :: y : x$ Do. do. do. in 16 months.

$\therefore 9 \times 8 : 24 \times 16 :: 120 : x$

$\therefore 3 \times 1 : 8 \times 2 :: 120 : x$

$\therefore 1 : 16 :: 40 : x; \therefore x = 640$.





5. If 248 men, in 5 days of 11 hours each, can dig a trench 230 yards long, 3 yards wide and 2 deep; in how many days of 9 hours length, will 24 men dig a trench of 420 yards long, 5 yards wide, and 3 deep?

Let x = days required.

Men.	Size of Trench.	Days.	Hours.
248	$230 \times 3 \times 2$ (=1380)	5	11
24	$420 \times 5 \times 3$ (=6300)	x	9

24 : 248 :: 5 : y , days of eleven hours length, in which 24 men would dig a trench of 1380 cubic yards.

1380 : 6300 :: y : z , days of eleven hours length, in which 24 men would dig a trench of 6300 cubic yards.

9 : 11 :: z : x , days of 9 hours length, in which 24 men would dig a trench of 6300 cubic yards.

$$\begin{aligned} \therefore 24 \times 1380 \times 9 & : 248 \times 6300 \times 11 :: 5 : x \\ \text{or } 3 \times 138 \times 1 & : 31 \times 70 \times 11 :: 5 : x \\ \text{or } 414 & : 23870 :: 5 : x \end{aligned}$$

$$\therefore x = \frac{119350}{414} = 28 \frac{118}{414} \quad 28 \frac{118}{414}$$

QUADRATIC EQUATIONS.

AN Equation, involving the square of the unknown quantity, (x) , is called a pure quadratic; as in the equation $x^2=9$. If it involve both the simple quantity (x) and the square of (x) , it is called an affected quadratic; as in the equation $x^2 + 8x = 20$.

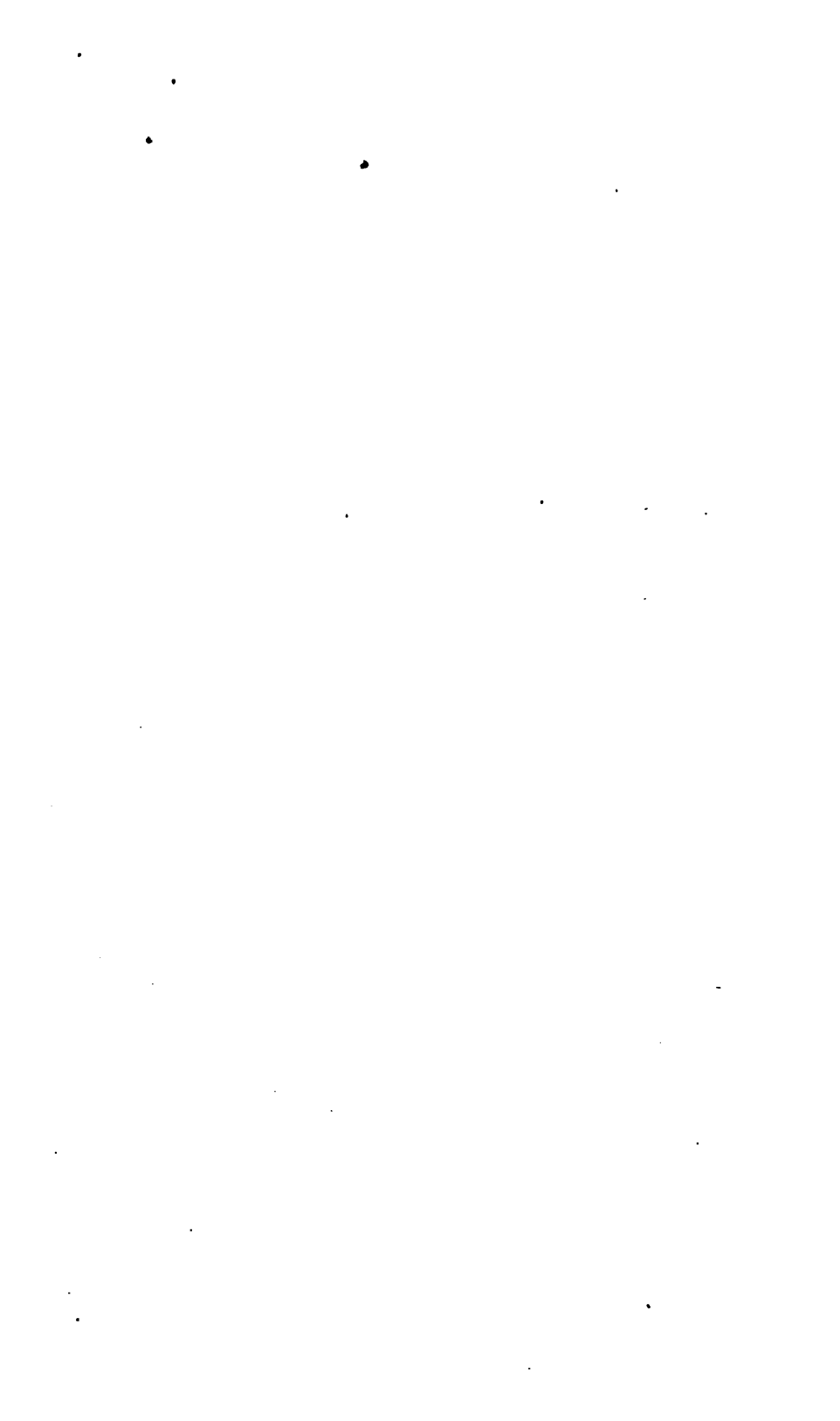
A pure quadratic is solved by extracting the square root of both sides of the equation. Thus, if $x^2 - 20 = 61$,
 $\therefore x^2 = 61 + 20 = 81 \therefore x = \pm 9$.

And in order to solve an affected quadratic, such a quantity is added to both sides of the equation as will make the first side the square of a binomial. Thus, if it be required to solve the equation $x^2 + 8x = 48$; since we know that the square of $(x + 4)$ is $x^2 + 8x + 16$, we make one side of the proposed equation agree with this trinomial, by adding 16 to it; we then have $x^2 + 8x + 16 = 48 + 16 = 64$, then extracting the square root of both sides, we have $x + 4 = \pm 8$, $\therefore x = \pm 8 - 4 = 4$ and -12 , the two values of x ; either of which when substituted for (x) in the original equation will answer the condition.

Again, if it be proposed to solve the equation $x^2 - 6x = -8$, since we know that $\overbrace{(x-3)}^2 = x^2 - 6x + 9$, make one side of the proposed equation, the same as this trinomial, by adding 9 to it; we then have $x^2 - 6x + 9 = 9 - 8 = 1$. Extract the square root, and $x - 3 = \pm 1$

$\therefore x = 3 \pm 1 = 4$ and 2 ;
 either of which numbers, when substituted for x in the proposed equation will satisfy the conditions.





It should be observed that in these solutions, in order to complete the square, we have merely added the square of half the co-efficient of (x) to both sides of the equation. The same method is adopted in all other cases. Thus, let it be required to solve $x^2 + 7x + 2 = 10$;

$$\text{then } x^2 + 7x = 10 - 2 = 8$$

$$\therefore x^2 + 7x + \frac{49}{4} = 8 + \frac{49}{4} = \frac{32+49}{4} = \frac{81}{4}$$

$$\therefore x + \frac{7}{2} = \pm \frac{9}{2}$$

$\therefore x = \pm \frac{9}{2} - \frac{7}{2} = 1$, and -8 ; the two values of x required.

It appears also that we extract the square root of the trinomial by taking the square root of x^2 , and adding to it half the co-efficient of x .

Given $3x^2 - 2x - 280 = 0$; to find x ;

$$3x^2 - 2x = 280.$$

Divide both sides by 3, that unity may be the co-efficient of x^2 ; then $x^2 - \frac{2x}{3} = \frac{280}{3}$

$$x^2 - \frac{2x}{3} + \frac{1}{9} = \frac{280}{3} + \frac{1}{9} = \frac{840+1}{9} = \frac{841}{9}$$

Then $x - \frac{1}{3} = \pm \frac{29}{3}$

$$\therefore x = \frac{1}{3} \pm \frac{29}{3} = 10 \text{ and } -\frac{28}{3}.$$

Given $x^2 - x = -\frac{2}{3}$

$$x^2 - x + \frac{1}{4} = \frac{1}{4} - \frac{2}{3} = \frac{1}{36}$$

$$x - \frac{1}{2} = \pm \frac{1}{6}$$

$$\therefore x = \frac{1}{2} \pm \frac{1}{6} = \frac{2}{3} \pm \frac{1}{6} = \frac{5}{6} \text{ and } +\frac{1}{3}.$$

Given $x + \sqrt{x} = 6$; this may be solved by either of the two following methods:

$$1. \sqrt{x} = 6 - x \therefore x = 36 - 12x + x^2$$

$$\therefore x^2 - 13x = -36 \therefore \&c.$$

$$\therefore x = 4 \text{ and } 9.$$

2. Or it may be solved by substitution. Thus,

Let $x = y^2 \therefore \sqrt{x} = y$

$$\therefore y^2 + y = 6$$

$$\therefore y^2 + y + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$$

$$\therefore y + \frac{1}{2} = \pm \frac{5}{2}$$

$$\therefore y = \pm \frac{5}{2} - \frac{1}{2} = 2, \text{ and } -3$$

$$\therefore x (=y^2) = 4 \text{ or } 9.$$

This mode of substitution may be adopted whenever the index of (x) in one term is half of what it is in the other. Thus, in the equation $x^2 - 4x^2 = 621$; let $x^2 = y \therefore x^4 = y^2$

$$\therefore y^2 - 4y = 621$$

$$y^2 - 4y + 4 = 621 + 4 = 625$$

$$y - 2 = \pm 25$$

$$\therefore y = 2 \pm 25 = 27 \text{ and } -23$$

$$\therefore x^2 (=y) = 27 \text{ and } -23$$

$$\therefore x = \sqrt[3]{27} \text{ and } \sqrt[3]{-23}.$$

Also in cases of binomial or multinomial, when the index of the proposed quantity in one term is half of what it is in another term. Thus, if $(2+x)^{\frac{1}{2}} - 8(2+x)^{\frac{1}{4}} = -12$; let $(2+x)^{\frac{1}{4}} = y \therefore (2+x)^{\frac{1}{2}} = y^2$.

$$\therefore y^2 - 8y = -12$$

$$y^2 - 8y + 16 = 16 - 12 = 4$$

$$y - 4 = \pm 2$$

$$\therefore y = 4 \pm 2 = 6 \text{ and } 2$$

$$\therefore (2+x)^{\frac{1}{4}} (=y) = 6 \text{ or } 2$$

$$\therefore 2 + x = 6^4 \text{ or } 2^4 = 1296 \text{ or } 16$$

$$\therefore x = 1296 - 2, \text{ or } 16 - 2.$$

1. Given $x^2 - 3x = 54$; to find the value of x . $x = 9 \text{ or } -6$

2. Given $2x^2 + 12x + 36 = 356$; to find x . $x = 10 \text{ or } -16$





3. Given $5x^2 - 24x = 5$; to find x . $= 5 \text{ or } -\frac{1}{5}$
4. Given $3x^2 + 2x = 161$; to find x . $= \pm \frac{\sqrt{1450} - 1}{3}$
5. Given $2x^2 - 5x = 117$; to find x . $= 9 \text{ or } -\frac{13}{2}$
6. Given $\frac{x^3}{3} - \frac{x}{2} = 9$; to find x . $= 6 \text{ or } -\frac{9}{2}$
7. Given $3x^3 + 4 = 19$; to find x . $= 25$
8. Given $\sqrt{ax+b^2} = a + b$; to find x . $= a + 2b$
9. Given $\sqrt{a} + \sqrt{x} = \sqrt{ax}$; to find x . $= \frac{a(a-2) \pm \sqrt{a(a-2)}}{(a-1)^2}$
10. Given $ax^2 - bx + c = d$; to find x . $= \frac{b \pm \sqrt{b^2 + 2a(d-c)}}{2a}$
11. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$; to find x . $= \frac{1 + \sqrt{2.00}}{2a}$
12. Given $\sqrt{5+x} - \sqrt{5-x} = 6$; to find x . $= 36 \text{ or } 11$
13. Given $\sqrt{1+x-x^2} - 2(1+x-x^2) = \frac{1}{9}$; to find x .
14. Given $x^{\frac{3}{2}} + x^{\frac{5}{2}} = 6x^{\frac{1}{2}}$; to find x .
15. Given $5x - \frac{3x-3}{x-3} = 2x + \frac{3x-6}{2}$; to find x .
16. Given $\frac{3}{x^2-3x} + \frac{6}{2x^2+8x} = \frac{27}{8x}$; to find x .
17. Given $\sqrt{4+x} \times \sqrt{5-x} = 2x-10$; to find x .
18. Given $4x - \frac{36-x}{x} = 46$; to find x .
19. Given $x + \frac{\sqrt{x-3}}{2} = 8$; to find x .
20. Given $x^{\frac{1}{2}} + 7x^{\frac{1}{2}} = 116$; to find x .
- Let $x^{\frac{1}{2}} = y$
- $\therefore x^{\frac{1}{2}} = y$, &c.
- 21 Given $\sqrt{y+a} = \sqrt{y} + \sqrt{b^2+y^2}$; to find y .
- Square both sides of the equation.
- $\therefore y+a = y + \sqrt{b^2+y^2}$
- $\therefore \sqrt{b^2+y^2} = a$; &c.

22. Given $\sqrt{x^3} - 2\sqrt{x} - x = 0$; to find x .

$$x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x = 0$$

Divide every term by $\sqrt{x}$; &c.

23. Given $x^{\frac{6}{7}} + x^{\frac{4}{7}} = 756$; to find x .

$$\text{Let } x^{\frac{4}{7}} = y.$$

$$\therefore y^2 + y = 756; \text{ \&c.}$$

24. Given $3x^{\frac{3}{5}} + x^{\frac{3}{10}} = 3104$; to find x .

$$\text{Let } x^{\frac{3}{5}} = y^2$$

$$y^2 + \frac{y}{3} = \frac{3104}{3}; \text{ \&c.}$$

25. Given $x + 3 - 7\sqrt{x+3} = 10 - 4\sqrt{x+3}$; to find x .

$$\text{Let } \sqrt{x+3} = y.$$

$$y^2 - 7y = 10 - 4y; \text{ \&c.}$$

26. Given $3x^2 + \sqrt{3x^2 + 4} = 16$; to find x .

Add 4 to both sides.

$$3x^2 + 4 + \sqrt{3x^2 + 4} = 16 + 4 = 20.$$

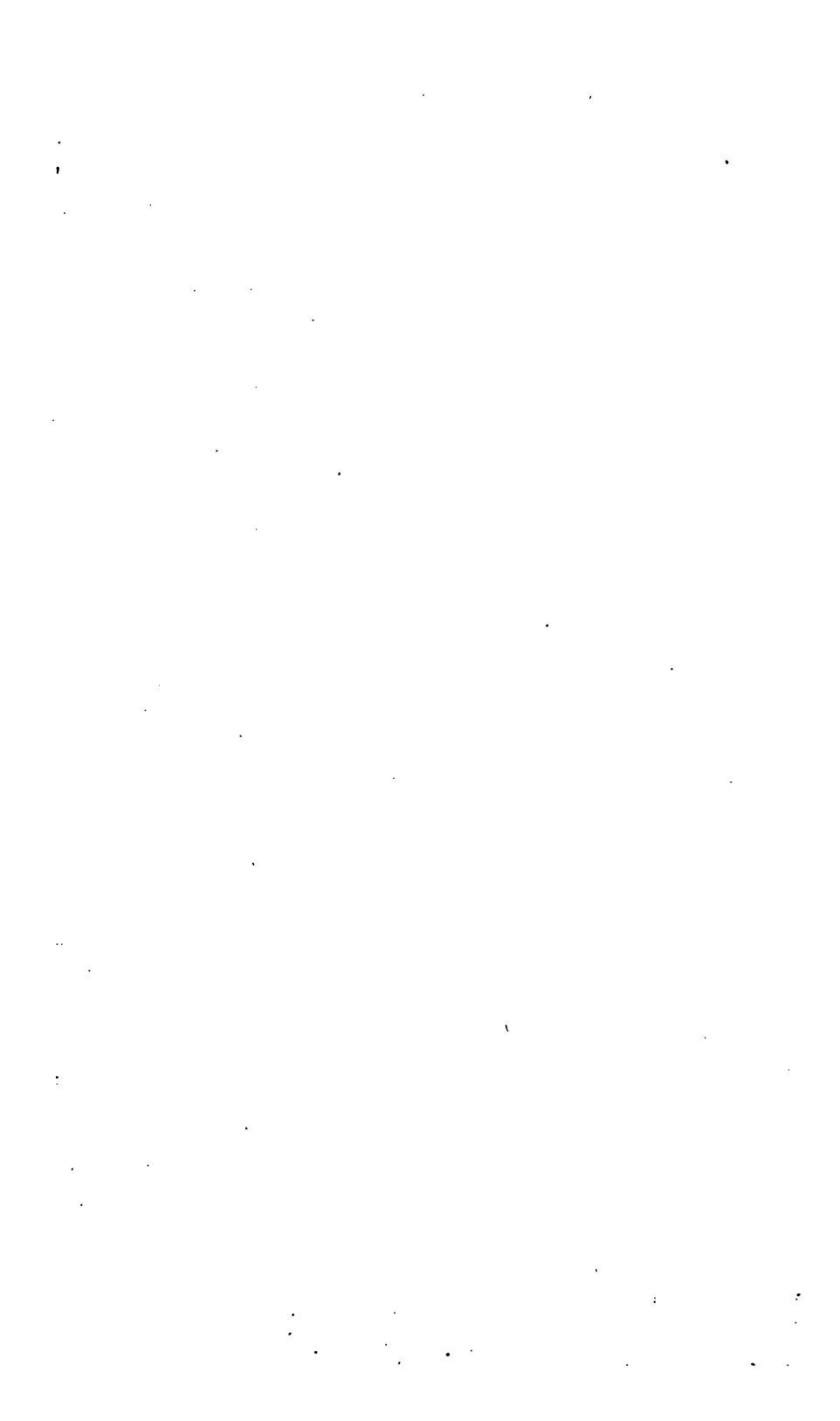
27. Given $\sqrt{4 + \sqrt{2x^3 + x^2}} = \frac{x+4}{2}$; to find x .

28. Given $\sqrt[3]{x^3 - a^3} = x - b$; to find x .

29. Given $\sqrt[4]{x+21} + \sqrt[3]{x+21} = 12$; to find x .

30. Given $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3$; to find x .





QUESTIONS

Producing *Quadratic* Equations.

Find two numbers, such that their difference may be 3, and their product 10.

Let x = the less number,

$\therefore x + 3$ = the greater,

$\therefore x \cdot x + 3 = 10$; &c. $x = 2$ and -5 .

The two numbers are 2 and 5; or -5 and -2 .

A person bought goods ~~for £56~~, and by selling them afterwards gained as much per cent. as the goods cost him; find the price he gave for the goods. *for 2*

Suppose the goods cost him x £,

Then he gained $56 - x$ by the sale;

$100 : x :: x : 56 - x$; &c. $x = 40$ and -140 .

The negative value of (x) can only apply to the case where he *received* £140 *with* the goods, instead of paying for them.

A party at a tavern have to pay a bill of £3. 10s. 0d.; four of them are defaulters, consequently each of the party has to pay 2s. more than his share; how many were there in company?

Let x = number of persons at first,

$\therefore \frac{70}{x}$ = each person's share of the bill,

$\frac{70}{x-4}$ = each person's share when four had left,

$\therefore \frac{70}{x-4} = \frac{70}{x} + 2$; &c. $x = 14$, and -10 .

The negative quantity (-10) applies to the case where the company have to *receive* money.

Find two numbers, of which the first is to the second as the second is to 36, and the difference of the squares of the numbers is 324.

Let x = first number,

$$\therefore x : \text{second} :: \text{second} : 36$$

$$\therefore \text{Second number} = \sqrt{36x}$$

$$36x - x^2 = 324$$

$$\therefore x^2 - 36x = -324, \text{ \&c.} \quad x = 18,$$

$$\therefore \sqrt{36x} = 18 \sqrt{2}.$$

A and B start at the same instant from two places, ~~100~~ 24 miles distant, and travel till they meet. A travels six miles a day, and the number of days at the end of which they meet is less by 2 than the number of miles B travels every day. How many miles did each go?

x = number of days they travel,

$$\therefore 6x = \text{do. miles A travels,}$$

$$x + 2 = \text{do. B travels every day,}$$

$$\therefore x \cdot \overline{x+2} = \text{do. B upon the whole,}$$

$$\therefore x^2 + 2x + 6x = \text{100, \&c.} \quad x = 6.$$

Given the sum of two numbers = 12, and the sum of their 4th powers = 3026, to find the numbers.

Let $6+x$ = greater of the two numbers,

$$\therefore 6-x = \text{less,}$$

$$\therefore \overline{6+x}^4 + \overline{6-x}^4 = 3026, \text{ \&c.} \quad x = 1.$$

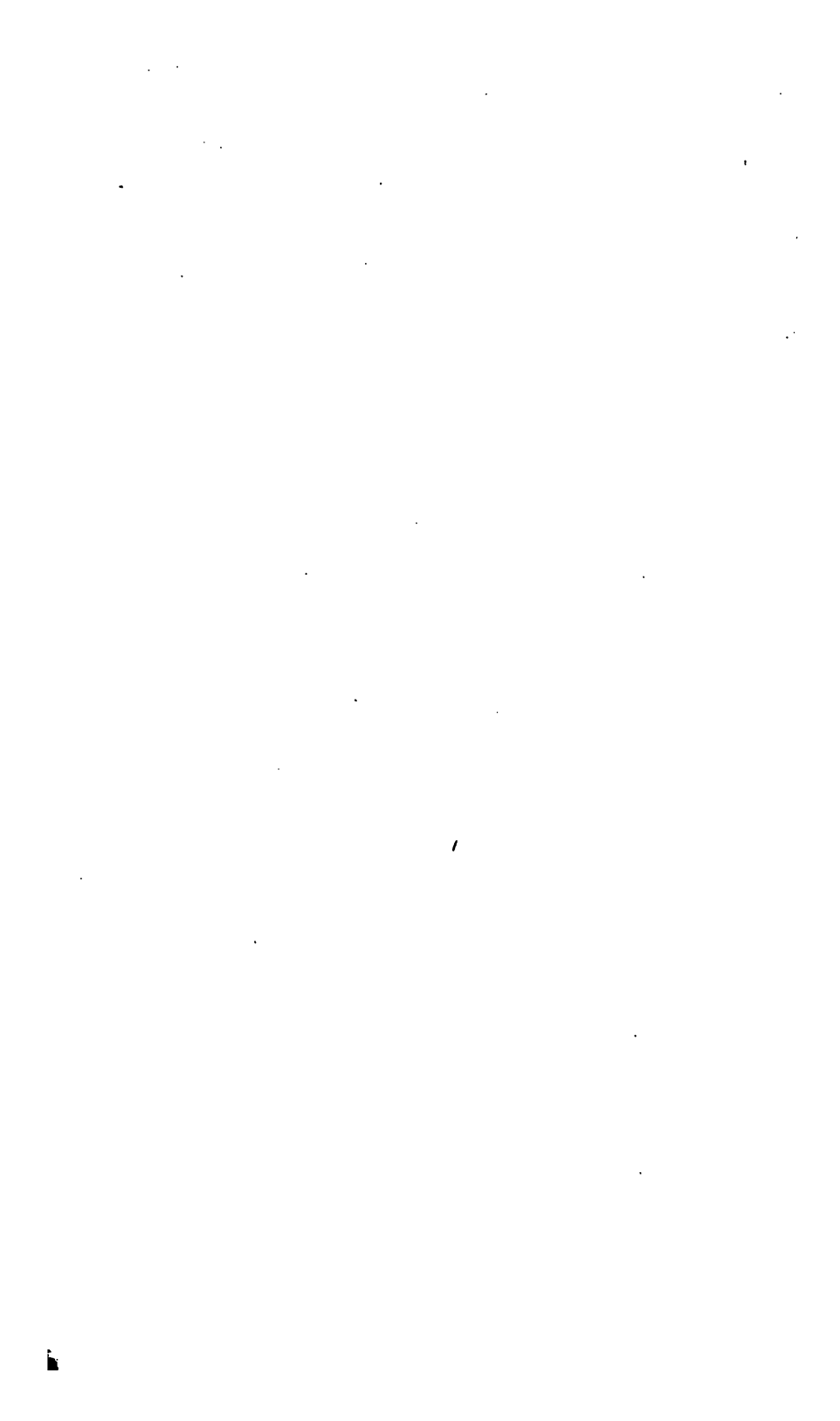
Divide the quantity (a) into two parts, so that their product may be = n^2 .

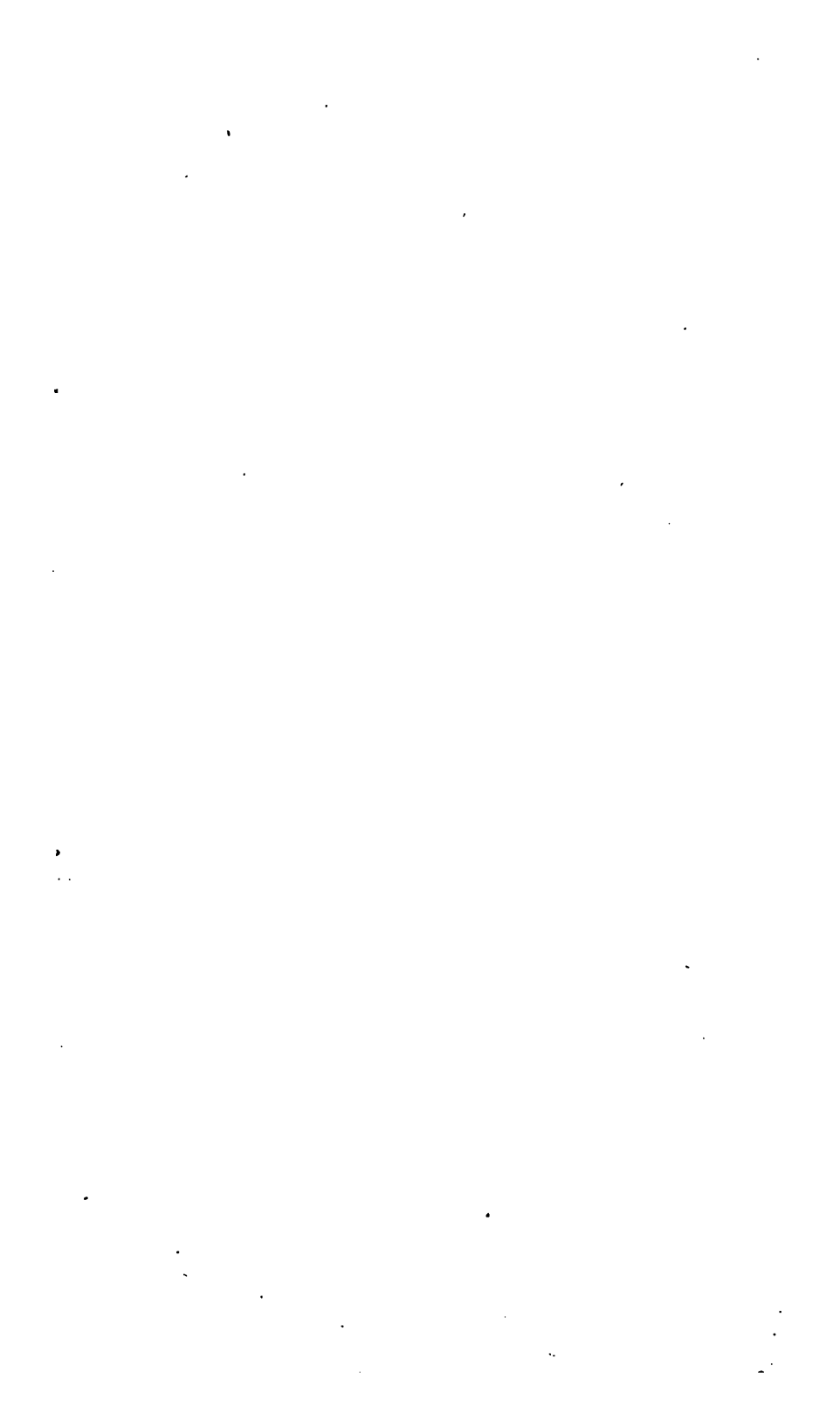
Let x = one part,

$$\therefore a-x = \text{the other,}$$

$$x \times \overline{a-x} = n^2, \text{ \&c.} \quad x = \frac{a \pm \sqrt{a^2 - 4n^2}}{2}$$

$$\therefore a-x = \frac{a \mp \sqrt{a^2 - 4n^2}}{2}$$





The given quantity (a) may be divided in various proportions, so that the product (n^2) of the two parts shall vary accordingly; but the greatest *possible* value of n^2 is evidently when $4n^2 = a^2$, or $n = \frac{a}{2}$, i.e. when (a) is bisected; for if (n) be greater than that, the value of x becomes impossible.

What is that number which being divided by the product of its two digits, gives a quotient = 2; and if 27 be added to the number, the digits will be inverted?

Let x and y be the digits,
then the number is $10x + y$

$$\therefore \frac{10x+y}{xy} = 2$$

$$\therefore 10x + y = 2xy$$

$$10x + y + 27 = 10y + x$$

$$9y - 9x = 27$$

$$\therefore y - x = 3$$

$$\therefore y = x + 3$$

$$\therefore 10x + x + 3 = 2x^2 + 6x$$

$$\therefore 2x^2 - 5x = 3, \&c.$$

$$x = 3$$

$$y = 6$$

$\therefore$ the number is 36.

Find two numbers such that their product shall be equal to the difference of their squares, and the sum of their squares be equal to the difference of their cubes.

Let x and y be the numbers,

$$\therefore xy = x^2 - y^2$$

$$x^2 + y^2 = x^3 - y^3$$

$$\therefore x^2 - 2xy + y^2 = x^3 - y^3 - 2(x^2 - y^2)$$

$$= x - y \cdot (x^2 + xy + y^2) - 2(x - y) \cdot (x + y)$$

$$\therefore x - y = x^2 + xy + y^2 - 2(x + y); \text{ but } xy = x^2 - y^2$$

$$\therefore x - y = 2x^2 - 2x - 2y$$

$$\therefore y = 2x^2 - 3x$$

$$\therefore 2x^3 - 3x^2 (=xy) = x^3 - y^3 = x^3 - (2x^2 - 3x)^2$$

$$\therefore 2x - 3 = 1 - (2x - 3)^2 = 1 - 4x^2 + 12x - 9, \text{ \&c.}$$

$$x = \frac{5 \pm \sqrt{5}}{4}$$

$$y = \frac{\pm \sqrt{5}}{2}$$

Find three numbers such that the difference of the two first shall be equal to the third; their sum equal to the square of the third; and their product equal to the cube of the third.

Let x , y , and z be the numbers,

$$\text{Then } x - y = z$$

$$x + y = z^2$$

$$xy = z^3$$

$$x - y = z$$

$$x + y = z^2$$

$$\therefore 2x = z^2 + z \quad \therefore x = \frac{z^2 + z}{2}$$

$$2y = z^2 - z \quad \therefore y = \frac{z^2 - z}{2}$$

$$\therefore 4xy = z^4 - z^2$$

$$\text{or, } 4z^3 = z^4 - z^2, \text{ \&c. } z = 2 \pm \sqrt{5}$$

$$x = \frac{11 \pm 5\sqrt{5}}{2}$$

$$y = \frac{7 \pm 2\sqrt{5}}{2}$$





INVOLUTION AND EVOLUTION.

INVOLUTION is the multiplying any Algebraic Quantity into itself a proposed number of times. Thus a^3 denotes the cube, or third power, of (a) ; the index shewing the number of times the quantity (a) is to be multiplied into itself; a^5 denotes $a \times a \times a \dots$ to five terms; a^n denotes $a \times a \times a \times \dots$ to (n) terms; $-a^5$ denotes $-a \times -a \times -a \times -a \times -a$, or $-\overbrace{a^5}^5$; $-a^5$ denotes $-a \times -a \times -a \times -a \times -a$, or $+a^5$. And, in general, if (n) be any integer, $-\overbrace{a^{2n}}^{2n}$ is the same as a^{2n} ; and $-\overbrace{a^{2n+1}}^{2n+1}$ is the same as $-a^{2n+1}$, because $(2n)$ is an even number, and $(2n+1)$ an odd number.

A quantity is raised from one power to a higher by multiplying together the exponents of those powers. Thus a^2 raised to the 4th power, becomes a^{12} ; for

$$a^2 = a \times a \times a$$

$$a^2 = a \times a \times a$$

$$a^2 = a \times a \times a$$

$$a^2 = a \times a \times a$$

Then $a^2 \times a^2 \times a^2 \times a^2 = a \times a \times a \dots$ to 12 terms.

Or a^2 to the 4th power $= a^{12}$

In like manner a^n raised to the m^{th} power is a^{mn} ; for $a^n = a \times a \times a \times a \dots$ to (n) terms; and a^n multiplied (m) times $= a \times a \times a \dots$ to (n) terms $\times a \times a \times a \times \dots$ to (n) terms $\dots (m)$ times repeated; which is the same as $a \times a \times a \times a \times a \dots$ to (mn) terms, or a^{mn} .

The product of two quantities may be raised to any power by raising each of them to that power. Thus $a^2 b^3$,

when raised to the fourth power, becomes a^4b^4 . The n^{th} power of a^3b^4 is $a^{3n}b^{4n}$.

A fraction is raised to any power by raising both numerator and denominator to that power. For $\frac{x}{y} \times \frac{x}{y} \times \frac{x}{y}$... to (n) terms is $\frac{x^n}{y^n}$.

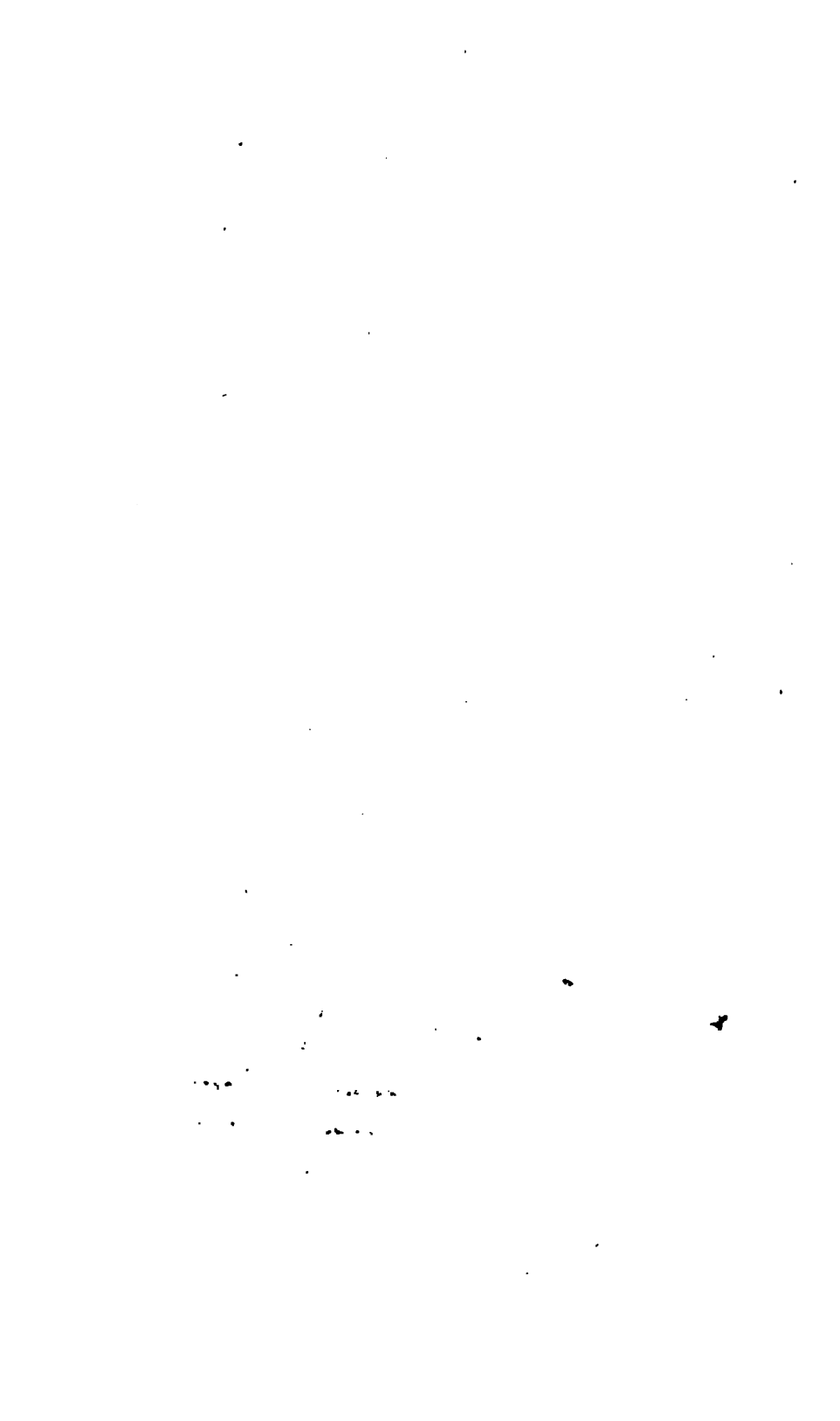
This rule applies to compound as well as simple quantities. Thus the n^{th} power of $(a+b)$ is $\overline{a+b}^n$; the m^{th} power of $\overline{a+b}^2$ is $\overline{a+b}^{2m}$; the n^{th} power of $\frac{\overline{a+b}^2}{x-y}$ is $\frac{\overline{a+b}^{2n}}{\overline{x-y}^{3n}}$.

The rule applies also to quantities with fractional indices. Thus the cube of $a^{\frac{1}{2}}$ is $a^{\frac{3}{2}}$ the n^{th} power of $a^{\frac{1}{m}}$ is $a^{\frac{n}{m}}$; the m^{th} power of $\overline{a+b}^{\frac{1}{3}}$ is $\overline{a+b}^{\frac{m}{3}}$.

The square of $\sqrt[3]{3} = 3^{\frac{2}{3}} = \sqrt[3]{9}$; the fourth power of $\overline{a+b}^{\frac{1}{5}} = \overline{a+b}^{\frac{4}{5}}$.

If the root denoted by the denominator of the fractional index cannot be extracted, the quantity is called a surd. Thus $\sqrt[3]{3}$, $\sqrt[3]{4}$, $\sqrt[3]{5}$, $\sqrt[3]{a^2}$, $\sqrt[3]{a^3}$, $\overline{a^3+b^3}^{\frac{1}{3}}$ are called surds.

If surds consist of different letters, but have the same fractional index, they may be multiplied together by placing their product under the common index. Thus, $a^{\frac{2}{n}} \times b^{\frac{2}{n}} = \overline{ab}^{\frac{2}{n}}$; for $a^{\frac{2}{n}} \times b^{\frac{2}{n}}$, raised to the n^{th} power is $a^{\frac{2n}{n}} \times b^{\frac{2n}{n}} = a^2 \times b^2$ or a^2b^2 ; and $\overline{ab}^{\frac{2}{n}}$ raised to the n^{th} power, is $\overline{ab}^{\frac{2n}{n}} = \overline{ab}^2$ or a^2b^2 . Thus $\sqrt[3]{4} \times \sqrt[3]{6} = \sqrt[3]{24}$; for the cube of $4^{\frac{1}{3}} \times 6^{\frac{1}{3}} = 4 \times 6$ or 24 ; and the cube of $24^{\frac{1}{3}}$ is also 24 .



+. Let $n=2$ an even number $\sqrt{-a}^0 = +a^2$
 which is a real & positive result. This formula is
 sufficient. The following is quite general.

If to $4n$, where n is any positive & real number

the result is positive & real.

$4n-2$, ————— negative & ^{real}imaginary.
 $4n+1$, ————— positive & imaginary
 $4n-1$, ————— negative & imaginary

Since the m^{th} power of $\sqrt[n]{a} = a^{\frac{m}{n}}$; let $n = m$; then the n^{th} power of $a^{\frac{1}{n}} = a$. Thus the cube of $\sqrt[3]{a^3 + x^3}$ is $a^3 + x^3$; the square of $\sqrt{a^2 \pm x^2}$ is $a^2 \pm x^2$; the 4th power of $\sqrt{a} \times \sqrt[4]{b^3 - y^3}$ is $a^3 (b^3 - y^3)$; the 5th power of $\sqrt[5]{a^5 \pm ax^4}$ is $a^5 \pm ax^4$.

Since then $\sqrt{a} \times \sqrt{b}$ is the same as $\sqrt{ab}$
 Suppose $b = -1$, $\therefore \sqrt{a \times -1}$ or $\sqrt{-a} = \sqrt{-1} \times \sqrt{a}$; (M)

$$\text{Then } \sqrt{-a}^2 = -1 \times a = -a; (M^2)$$

$$\therefore \sqrt{-a}^3 = -a\sqrt{-a}; (M^2 \times M)$$

$$\therefore \sqrt{-a}^4 = +a^2; (M^2 \times M^2)$$

$$\sqrt{-a}^5 = a^2 \sqrt{-a}; (M^4 \times M)$$

$$\sqrt{-a}^6 = -a^3; (M^4 \times M^2)$$

$$\sqrt{-a}^7 = -a^3 \sqrt{-a}; (M^6 \times M)$$

$$\sqrt{-a}^8 = +a^4; (M^6 \times M^2)$$

Thus it appears that if the imaginary quantity $\sqrt{-a}$ be raised to the power $(4n)$, where (n) is an odd number, $\times$ the result is real and positive; if it be raised to the power $(2n)$ the result is real and negative. "If it be raised to the power $(2n+1)$, the result is a negative imaginary quantity; if to the power $(4n+1)$ the result is a positive imaginary quantity."

If we raise $(a+b)$ to the 5th power, we find $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$; where it appears that the co-efficients, equidistant from each end of the series, are the same; and the indices of (a) successively decrease, while those of (b) increase, by unity. The co-

efficient too, in any term, is found from multiplying together the index of (a) and the co-efficient in the preceding term, and dividing their product by the number of preceding terms. Thus to find the co-efficient in the third term, we multiply 5 and 4 together, and divide this product by 2, the number of preceding terms. The whole number of terms is 6, greater by unity than the index of the proposed binomial. These remarks apply to every case. If then I wish to raise $(a+b)$ to the 6th power, I write down the terms without the co-efficients thus, a^6 , a^5b , a^4b^2 , a^3b^3 , a^2b^4 , ab^5 , b^6 . And to find the co-efficients, I prefix unity to the first term; the co-efficient of the 2d term is 6, the index of the binomial; the co-efficient of the 3d term is $\frac{5 \times 6}{2} = 15$; the co-efficient of the 4th

term is $\frac{15 \times 4}{3} = 20$. The first 4 terms then will be

$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3$; and as there will be, upon the whole, 7 terms, and the co-efficients of the corresponding terms are the same, I know that the remaining terms must be these, $15a^2b^4 + 6ab^5 + b^6$.

Raise $(a+b)$ to the third power.

Raise $(a+b)$ to the fourth power.

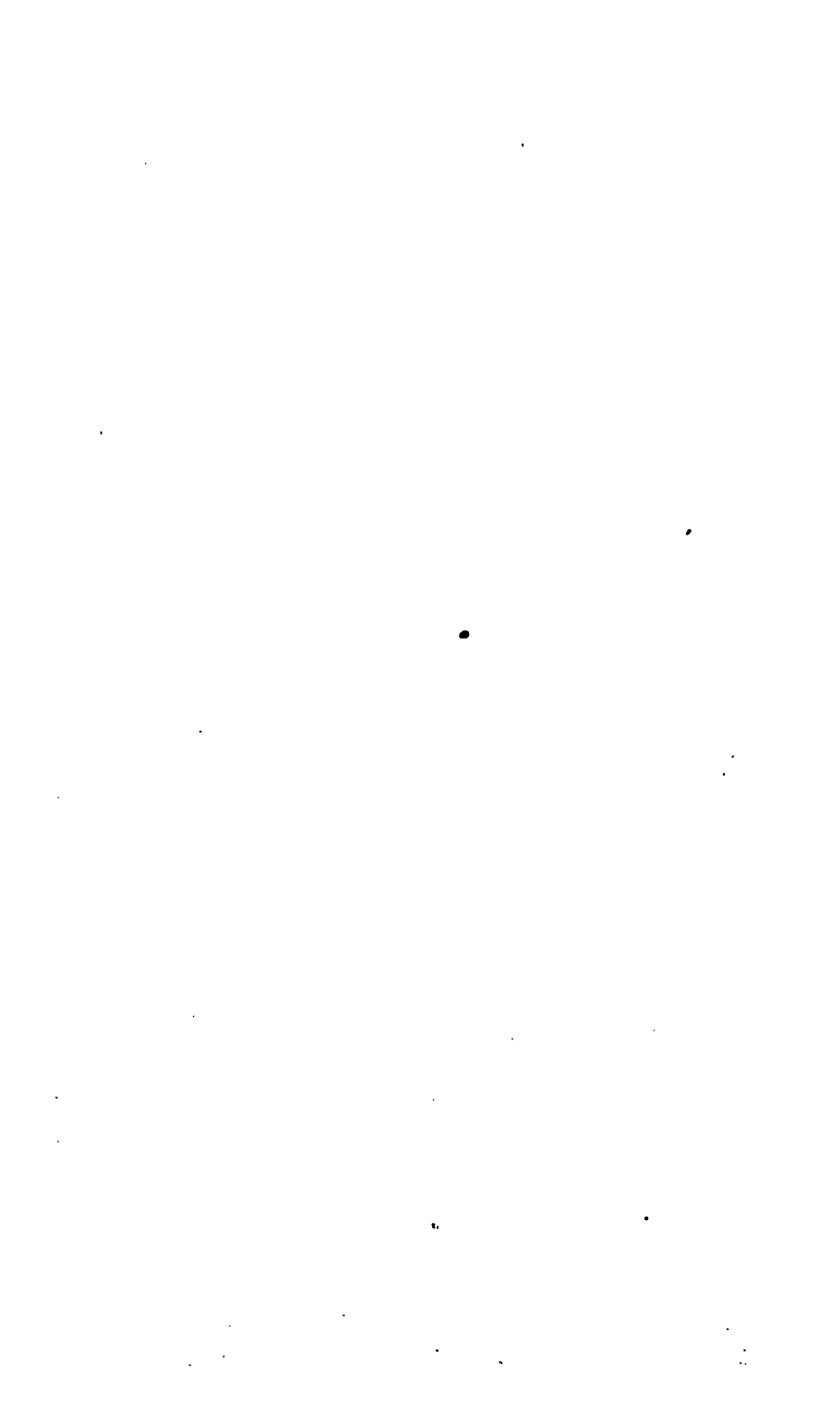
Raise $(a-b)$ to the fourth power.

Raise $(a+b)$ to the seventh power.

Find the 6th power of $2x+y$.

Find the 5th power of $y-2x$.





EVOLUTION,

Or the Extracting of Roots, is the reverse of Involution, and is performed by dividing the index of the proposed quantity by the number expressing the required root. Thus if the cube of a^2 be a^6 , the cube root of a^6 must be a^2 , or $a^{\frac{6}{3}}$. If (a) be assumed $= 2$, $a^2 = 4$, and the cube of 4 is 64: the cube root then of 64 or of 4^3 is 4, or $4^{\frac{3}{3}}$. The cube root of a^2x^6 is $a^{\frac{2}{3}}x^2$; the 4th root of a^2x^8 is $a^{\frac{1}{2}}x^{\frac{2}{2}}$; the n^{th} root of $a^m b^p$ is $a^{\frac{m}{n}} b^{\frac{p}{n}}$.

The roots of a fraction are found by extracting the root of both numerator and denominator. For since $\frac{a}{b}$, when raised to the n^{th} power, is $\frac{a^n}{b^n}$ the n^{th} root of $\frac{a^n}{b^n}$ is $\frac{a}{b}$ or $\frac{a^{\frac{n}{n}}}{b^{\frac{n}{n}}}$.

Find the square root of $\frac{a^4 b^6}{x^8}$.

Find the square root of $\frac{a+4b}{x-3y}$.

Find the cube root of $\frac{2a+3b}{x+y}$.

Find the 4th root of $\frac{3x+2y}{a-3b}$.

Since a^{-m} is the same as $\frac{1}{a^m}$ the n^{th} root of a^{-m} is the n^{th} root of $\frac{1}{a^m}$, and is $\frac{1}{a^{\frac{m}{n}}} = a^{-\frac{m}{n}}$.

From observing that the square root of $a^2 + 2ab + b^2$ is $a + b$; the following rule has been given for finding the *square root of a Compound Quantity*. Arrange the terms according to the dimensions of some letter; extract the root of the first term; place this root in the quotient; subtract its square from the given quantity; to twice this quotient annex the quotient next found, for a divisor; multiply the whole of this divisor by the second term of the quotient; subtract this product from the remaining terms of the proposed quantity; and repeat this process till there be no remainder. Thus, in extracting the square root of $a^2 + 2ab + b^2$,

$$\begin{array}{r}
 a^2 + 2ab + b^2 \ (a + b \\
 \underline{a^2} \\
 2a + b \) \ 2ab + b^2 \\
 \underline{2ab + b^2} \\
 * \quad *
 \end{array}$$

If we assume $a=12$, and $b=9$, the proposed number is 441, or $144 + 216 + 81$; then, according to the above rule,

$$\begin{array}{r}
 144 + 216 + 81 \ (12 + 9 \\
 \underline{144} \\
 24 + 9 \) \ 216 + 81 \\
 \underline{216 + 81} \\
 * \quad *
 \end{array}$$

And the square root is $12 + 9$ or 21, the same result as if we adopt the common method.

$$\begin{array}{r}
 \text{Thus } 400 + 40 + 1 \ (20 + 1 \\
 \underline{400}
 \end{array}$$

$$\begin{array}{r}
 40 + 1 \) \ 40 + 1 \\
 \underline{40 + 1} \\
 * \quad *
 \end{array}$$

$$\begin{array}{r}
 \text{Or thus } 441 \ (21 \\
 \underline{4}
 \end{array}$$

$$\begin{array}{r}
 41 \) \ 41 \\
 \underline{41} \\
 *
 \end{array}$$





digits. Hence arises the rule, in Arithmetic, for pointing off the proposed square number into periods of two digits.

Extract the square root of $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$
 $a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \ (a + b + c$
 a^2

$2a + b \) \ 2ab + b^2$
 $2ab + b^2$

$2a + 2b + c \) \ + 2ac + 2bc + c^2$
 $2ac + 2bc + c^2$

* * *

Extract the square root of $x^4 - 4x^3 + 6x^2 - 4x + 1$.

Extract the square root of $4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4$.

Extract the square root of $4x^4 - 16x^3 + 24x^2 - 16x + 4$.

Extract the square root of $16a^3 + 24a^2 + 89a^2 + 60a + 100$.

Extract the square root of $9y^5 - 12y^4 - 10y^4 - 28y^3 + 17y^2 - 8x + 16$.

Extract the square root of $1 + x$.

Extract the square root of $4a^5 - 16a^4 - 16a^3 + 12a^2 + 32a^3 + 24a^2 + 8a + 1$.

Extract the square root of $a^2x^3 + a^2$.

Extract the square root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16}$

Extract the square root of $x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16$.

Extract the square root of $a^2 + b$; and of 2, or of $1 + 1$.

The rule for finding the cube root of a compound quantity is derived from observing in what manner the cube root $(a + b)$ of $a^3 + 3a^2b + 3ab^2 + b^3$ may be extracted.





Arrange the terms (as in the square root) according to the dimensions of some letter; place the root (a) of the first term in the quotient; subtract its cube from the proposed quantity; divide the first term ($3a^2b$) of the remainder by 3 times the square of the root (a), and the quotient is the second term in the root; subtract the cube of the root thus found from the given quantity. Again, divide the first term of the remainder by 3 times the square of the first term, as before, and the quotient is the third term in the root.

$$\begin{array}{r}
 a^3 + 3a^2b + 3ab^2 + b^3 \quad (a + b \\
 \underline{a^3} \\
 3a^2b \quad \text{first term of rem.} \\
 \underline{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3} \\
 * \quad * \quad * \quad *
 \end{array}$$

EXAMPLE.

Find the cube root of $a^3 + 3a^2c + 3ac^2 + c^3 + 3a^2d + 6acd + 3cd^2 + 3ad^2 + 3cd^2 + d^3$.

$$\begin{array}{r}
 a^3 + 3a^2c + 3ac^2 + c^3 + 3a^2d + 6acd + 3cd^2 + 3ad^2 + 3cd^2 + d^3 \\
 \underline{a} \qquad \qquad \qquad [a + c + d
 \end{array}$$

$$\begin{array}{r}
 3a^2 \quad 3a^2c \quad (\text{first term of remainder}) \\
 \underline{a^3 + 3a^2c + 3ac^2 + c^3}
 \end{array}$$

$$3a^2 \quad 3a^2d \quad (\text{first term of remainder})$$

Now the cube of $(a + c + d)$ subtracted from the proposed quantity, leaves no remainder; consequently $\underline{a + b + c}$ is the root required. $a +$

Extract the cube root of $8x^3 + 36x^2 + 54x + 27$

$$\begin{array}{r}
 8x^3 + 36x^2 + 54x + 27 \quad (2x + 3 \\
 \underline{8x^3}
 \end{array}$$

$$12x^2 \quad 36x^2 \quad (\text{first term of remainder})$$

Now as the cube of $(2x+3)$ is equal to the proposed quantity, $2x+3$ is the root required.

Extract the cube root of $27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1$.

$$\begin{array}{r} 27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1 \quad (3x^2 - 2x + 1 \\ \underline{27x^6} \end{array}$$

$27x^4) - 54x^5$ (first term of remainder)

$(3x^2 - 2x)^3 = 27x^6 - 54x^5 + 36x^4 - 8x^3$; and this subtracted from the original quantity gives $27x^4 - 54x^3 + 21x^2 - 6x + 1$; the first term of this quantity divided by $27x^4$ gives quotient (1); and the cube of $(3x^2 - 2x + 1)$ is $27x^6 - 54x^5 + 21x^4 - 6x^3 + 1 \therefore 3x^2 - 2x + 1$ is the root required.

Find the cube root of $x^6 - 6x^4 - 40x^3 + 96x - 64$.

Find the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

This method is easily applied to numbers. Thus to extract the cube root of 614125.

$$\begin{array}{r} 614125 \quad (85 \\ \underline{512} \end{array}$$

$$3 \times 8^2 = 192 \quad) \quad 102125 \quad ($$

Now the cube of $85 = 614125 \therefore 85$ is the root required.

For practical purposes, however, it may be more convenient to use the following simple method of extracting the cube root in numbers. *First* then, it must be observed, that when the unit digit of the proposed cube number is 2, the unit digit of the cube root is 8, and *vice versa*. If the unit digit of the proposed cube be 3, the unit digit of the cube root is 7, and *vice versa*. In all other numbers, the unit digit of the cube, and of the cube root, is the same.





Secondly, if the cube consist of 3 digits, the cube root consists of one digit. Suppose then it were required to find the cube root of 343. Since the unit digit of the cube is 3, the unit digit of the cube root is 7; and as the proposed cube consists of only 3 digits, the cube root consists of only one digit, $\therefore$ 7 is the root required.

If it be required to extract the cube root of 2744. First, we know that the cube root consists of two digits; of which the unit digit is 4, because the unit digit of the proposed cube is 4; and if we point off in the proposed number a period of 3 digits, we find the first digit of the root to be 1; $\therefore$ the root required is 14. If it be required to find the cube root of 614125; point off the digits into periods of three. Find, by trial, the next inferior cube root to that of 614; this is 8; the next digit in the root is 5, because the unit digit of the proposed number is 5; $\therefore$ the root required is 85. If it be required to find the cube root of 41421736; we find the next inferior cube root to that of 41 to be 3. We then need only proceed to find the second figure in the root, which is 4; and we know, from the unit digit in the proposed number, that the unit digit in the root must be 6; $\therefore$ the root required is 346.

In extracting the cube root, in Arithmetic, periods of three digits are pointed off, because a number consists of $(3n)$ or $(3n-1)$ or $(3n-2)$ digits, when its cube root consists of (n) digits.

The square root of a quantity cannot be partly rational and partly irrational; for, if possible, let $\sqrt{a} = b + \sqrt{c}$
 $\therefore a = b^2 + 2b\sqrt{c} + c \therefore \sqrt{c} = \frac{a - b^2 - c}{2b}$; a rational quantity equal to an irrational one; which is impossible.

Hence we infer that if $a + \sqrt{x} = b + \sqrt{y}$, where a and b are rational quantities, and $\sqrt{x}$ and $\sqrt{y}$ are quadratic surds, $a = b$, and $\sqrt{x} = \sqrt{y}$. For if in the equation, $\sqrt{x} = b - a + \sqrt{y}$, $\sqrt{x}$ be not equal to $\sqrt{y}$, it must be equal to a quantity partly rational and partly irrational, which has been shewn to be impossible.





GREATEST COMMON MEASURE

AND

LEAST COMMON MULTIPLE.

Definition.

ONE quantity is said to measure another when it is contained in it a certain number of times exactly. A common measure of two or more quantities is a quantity which divides them without leaving a remainder; and the *greatest common measure* is the greatest quantity by which they can be so divided.

If one quantity (a) measure two others, b and c , it measures their sum and difference. For, let $na = b$, and $ma = c$, $\therefore b \pm c = ma \pm na = \overline{m \pm n} \cdot a$, an integral quantity, since (m) and (n) are integers. Thus 8 is a measure of 48 and of 64; it also measures 112 and 16, their sum and difference; 6 measures 42 and 54; it also measures 96 and 12.

The rule for finding the greatest common measure of two quantities is, to “divide the greater by the less; the first divisor by the first remainder; the second divisor by the second remainder, and so on till there be no remainder; the last divisor is the greatest common measure. Thus to find the greatest common measure of (a and (b);”

$ \begin{array}{r} b) a (p \\ \underline{pb} \\ c) b (q \\ \underline{qc} \\ d) c (r \\ \underline{dr} \\ * * \end{array} $	Let (b) be contained in (a) p times, and leave a remainder (c); let (c) be contained in (b) q times, and leave a remainder (d); let (d) be contained in (c) r times. Then $a = pb + c$ and $a - pb = c$ $b = qc + d$ $b - qc = d$ $c = rd$ $c = dr$ * d is the greatest common measure of (a) and (b).
---	---

First it is a measure of (a) and (b) : for since (d) measures (c) , it measures qc , and therefore $qc + d$ or b : it therefore measures pb ; and $pb + c$ or a . It is also the *greatest common measure* of a and b ; for *every* other common measure of (a) and (b) is a measure of $a - pb$ or c : also any quantity that measures (b) and (c) measures $b - qc$ or (d) ; $\therefore (d)$ is the greatest common measure of (a) and (b) .

Find the greatest common measure of 234 and 3348.

$$\begin{array}{r}
 234 \overline{) 3348} (14 \\
 \underline{234} \\
 1008 \\
 \underline{936} \\
 72 \overline{) 234} (3 \\
 \underline{216} \\
 18 \overline{) 72} (4 \\
 \underline{72} \\
 *
 \end{array}$$

18 is the greatest common measure of the proposed numbers.

* If d is not the greatest common measure
let x be greater & let $a = mx$ & $b = nx$

$$\therefore c = a - pb = mx - pnx = m - pn \cdot x$$

$$d = b - qc = nx - qm - pqn \cdot x =$$

$$= n - qm + pqn \cdot x$$

or x measures d , which is impossible $\therefore$ & c.



Find the greatest common measure of x^4-1 and x^3+x .

$$\begin{array}{r}
 x^3+x \) \ x^4-1 \ (\ x \\
 \underline{x^4+x^2} \\
 -x^2-1 \\
 \text{or } -1+(x^2+1) \) \ x^3+x \ (\ +x \\
 \underline{x^3+x} \\
 * \quad *
 \end{array}$$

$\therefore x^2+1$ is the greatest common measure.

Or, since the quantity x^3+x is the same as $x \times (x^2+1)$;
divide x^4-1 by x^2+1

$$\begin{array}{r}
 x^2+1 \) \ x^4-1 \ (\ x^2 \\
 \underline{x^4+x^2} \\
 -x^2-1 \\
 \text{or } -1 \times (x^2+1)
 \end{array}$$

Then x^2+1 is the greatest common measure, as before.

Find the greatest common measure of $\frac{x^3-b^2x}{x^3+2bx+b^2}$

The numerator is the same as $x(x^2-b^2)$; expunge x .

$$\begin{array}{r}
 x^2-b^2 \) \ x^3+2bx+b^2 \ (\ 1 \\
 \underline{x^3-b^2} \\
 2bx+2b^2 \text{ or } 2b \times (x+b); \text{ expunge } 2b. \\
 \underline{x+b \) \ x^2-b^2 \ (\ x-b \\
 \underline{x^2-b^2} \\
 * \quad *
 \end{array}$$

and the greatest common measure is $x+b$.

Find the greatest common measure of $14x^2-7xy$ and $10ax-5ay$.

$$\begin{array}{l}
 14x^2-7xy = 7x(2x-y) \\
 10ax-5ay = 5a(2x-y)
 \end{array}$$

From which it is manifest that $2x-y$ is the greatest common measure.

Find the greatest common measure of $4ax + 5bx - 6x^2$ and $12ayz + 15byz - 18xyz$.

The first quantity is $x(4a + 5b - 6x)$

The latter quantity is $3yz(4a + 5b - 6x)$

And it is clear that $4a + 5b - 6x$ is the greatest common measure.

Find the greatest common measure of $2a^3 + 3a^2x - 9ax^2$ and $6a^3x - 17a^2x^2 + 14ax^3 - 3x^4$.

The first quantity is $a(2a^2 + 3ax - 9x^2)$

The latter quantity is $x(6a^3 - 17a^2x + 14ax^2 - 3x^3)$

$$2a^3 + 3ax - 9x^2 \begin{pmatrix} 6a^3 - 17a^2x + 14ax^2 - 3x^3 \\ 6a^3 + 9a^2x - 27ax^2 \end{pmatrix} \begin{pmatrix} 3a - 13x \end{pmatrix}$$

$$-26a^2x + 41ax^2 - 3x^3$$

$$-26a^2x - 39ax^2 + 117x^3$$

$$80ax^2 - 120x^3$$

$$\text{or } 40x^2(2a - 3x)$$

$$(2a - 3x) \begin{pmatrix} 2a^2 + 3ax - 9x^2 \\ 2a^2 - 3ax \end{pmatrix} \begin{pmatrix} a + 3x \end{pmatrix}$$

$$2a^2 - 3ax$$

$$6ax - 9x^2$$

$$6ax - 9x^2$$

* *

$\therefore 2a - 3x$ is the greatest common measure.

To find the greatest common measure of three quantities, a , b , and c ; first find (m) the greatest common measure of (a) and (b) ; then find the greatest common measure of (m) and c , and this will be the greatest common measure of a , b , and c . For since (m) is the greatest common measure of (a) and (b) , every quantity that measures (m) must measure (a) and (b) ; therefore every common measure of

The most general proof of the greatest common measure or highest common divisor of two algebraical quantities as follows.

$$\begin{array}{r}
 B \div A \text{ (P)} \\
 \underline{PB} \\
 Cc \\
 \text{divide by } c \quad C \div B \text{ (Q)} \\
 \underline{QC} \\
 Dd \\
 \text{divide by } d \quad D \div C \text{ (R)} \\
 \underline{RD} \\
 \dots
 \end{array}$$

c and d are quantities which are not divisors of A and B and which have no factor which is common then. To prove this ^{operation} we must establish the following

Lemma. If D is the highest common divisor of A & B it is likewise the hight. com: div. of Aa and Bb , if a and b have no common factors. For the factors which are common to A & B are the same which are common to Aa and Bb no new common factor having been introduced by multiplying a into A and b into B . It follows from this Lemma, that we may divide, multiply either A or B , by a factor which is not common to both or which has no divisor which is common to both, without affecting the dimensions or form of the highest common factor.

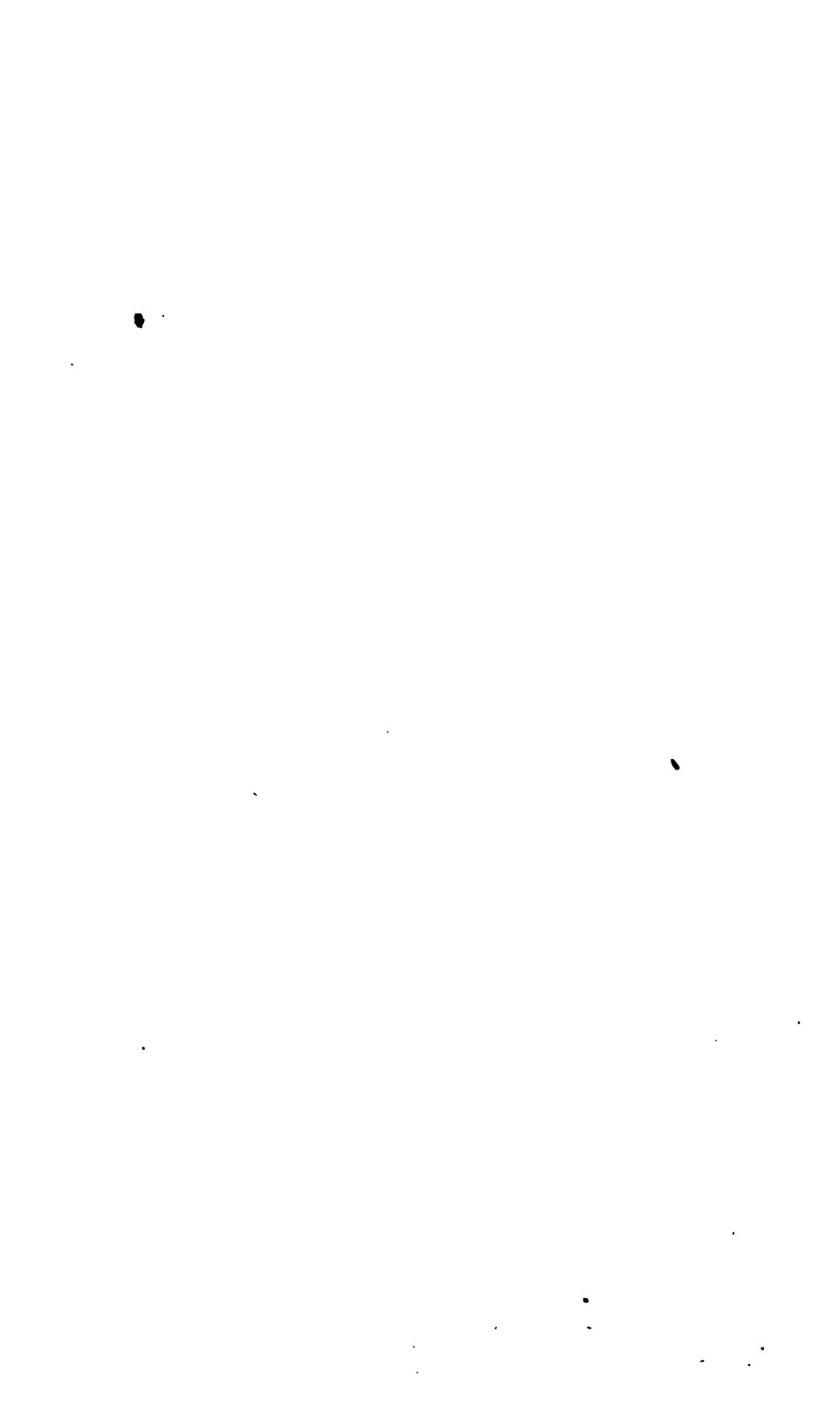
D is a common factor of A & B ; for since

$$C = RD; B = QC + Dd; A = PB + Ce$$

D is a divisor of C & $\therefore$ of QC and D , & $\therefore$ of $QC + Dd$ or B , & $\therefore$ of PB , & $\therefore$ of $PB + Ce$ or A .

It is the greatest; for

Every other factor of A and B is a factor of $A - PB$, of $A - PB$ or Ce , & $\therefore$ of C alone (since e has no factor in common with A & B), & $\therefore$ of $B - QC$, & $\therefore$ of $B - QC$ or Dd ; & $\therefore$ of D alone, since d has no factor in common with A & B .



(m) and (c) is a common measure of a , b , and c ; $\therefore$ the greatest common measure of (m) and (c) is the greatest common measure of a , b , & c . The same mode of reasoning applies to any number of quantities whatever.

LEAST COMMON MULTIPLE.

The least common multiple of two quantities, is the least quantity which they will both divide without a remainder; and it is found by dividing the product of the quantities by their greatest common measure. Thus, since the greatest common measure of 210 and 360 is 30, the least common multiple of these numbers is $\frac{210 \times 360}{30} = 2520$.

Let A and B be two given quantities, and suppose the ratio of $A : B$ to be, in its *lowest terms* $:: m : n$; it is evident then that $nA (=mB)$ is the *least common multiple* of A and B . And since (m) and (n) are the least possible integers in the ratio of $A : B$, it follows that if $c =$ greatest common measure of A and B , $\frac{A}{c} = m$, and $\frac{B}{c} = n$, $\therefore \frac{BA}{c} = (Bm \text{ or } An)$ is the least common multiple.

To find the least common multiple of a , b , & c .

Suppose (M) to be the least common multiple of (a) and (b), every multiple then of (M) is a common multiple of (a) and (b); and every common multiple of (M) and (c) is a common multiple of a , b , and c ; $\therefore$ the least common

multiple of (M) and (c) is the least common multiple of a , b , and c .

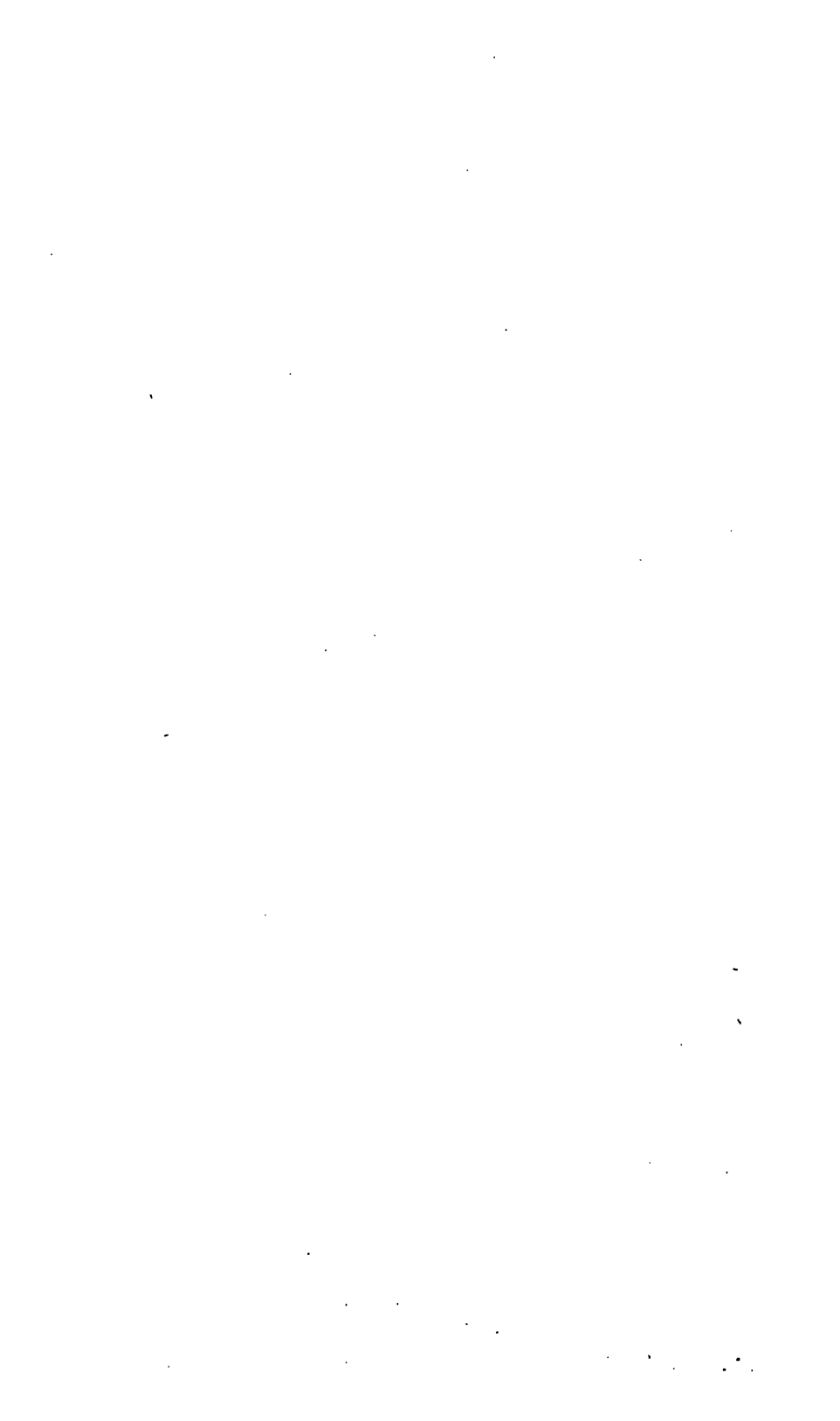
Find the least common multiple of $x^2 + 1$ and $x^2 + 2x + 1$.

$$\begin{array}{l} -x/ \\ x^2 + 1 = (x+1) \cdot (x^2 \cancel{x} + 1) \\ x^2 + 2x + 1 = (x+1) \cdot (x+1) \end{array}$$

And the greatest common measure of the two quantities is $x+1$;

$$\begin{array}{l} x/ \\ -x/ \\ \therefore \text{the least common multiple is } \frac{(x+1)^2 \times (x^2 \cancel{x} + 1)}{x+1} \\ = \overbrace{x+1}^2 \times (x^2 \cancel{x} + 1). = x^4 + x^3 + x + 1 \end{array}$$





PERMUTATIONS AND COMBINATIONS.

Definition.

1. *Permutations* are the various orders in which any collection of quantities may be placed. *Combinations* are the collections themselves, without regarding the order of their arrangement. For instance, a combination of three quantities, a, b, c , admits of six permutations, $abc, acb, bac, bca, cab, cba$.

2. The number of permutations which two quantities, c, d , admit, is evidently two, cd, dc . If I join (b) to them, and form all possible permutations out of the three, it is evident that (b) may be placed either *before* or *after* each of the quantities in the two preceding permutations; and may therefore have *three* different places for each of these two permutations; consequently there must be 3×2 permutations of the three quantities, b, c, d . Again, if I join (a) to these three quantities, b, c, d , it is also clear that, by placing (a) either *before* or *after* any quantity in each of the six preceding permutations, it may have four different places; $\therefore$ the number of permutations of four quantities, a, b, c, d , taken together, is $4 \times 3 \times 2$ or 24. In like manner the number of permutations of 7 quantities will be $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$; and, in general, a combination of (n) quantities taken together, admits of $1.2.3. \dots (n)$ permutations.

EXAMPLES

1. How many changes can be rung upon 5 bells?

$$5 \times 4 \times 3 \times 2 \times 1 = 120. \text{ Answer.}$$

2. In how many different ways can the letters in the word "Eton" be written?

$$4 \times 3 \times 2 \times 1 = 24. \text{ Answer.}$$

3. In how many ways can 7 persons vary their position round a table?

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040. \text{ Answer.}$$

Since then it appears that every combination of 2 quantities admits of 2×1 permutations, and every combination of 3 quantities admits of $3 \times 2 \times 1$ permutations, it is evident that if 2 out of 4 quantities become equal to each other, the number of permutations is diminished by half. If three become equal, the number of permutations is $\frac{1}{3 \times 2}$ of the original number.

Thus the number of permutations of 4 quantities, a, b, c, d , taken all together is $4 \times 3 \times 2 \times 1 = 24$; but if (b) become (a), then there can be but one order of placing ab , or ba , ($=aa$); therefore this combination, which did admit of two permutations, now admits of one only.

Again, suppose c and b to become each $= a$, then the combination a, b, c , which did admit of 3×2 permutations, now admits of one only, as abc is become aaa . Therefore the original number (24) of permutations made out of 4 quantities, must become $\frac{24}{3 \times 2}$, when 3 of them are equal: For instance, the quantities are a, a, a, d , and all the permutations they admit are $aaad, aada, adaa, daaa$ which $= \frac{24}{6} = 4$; and, in general, the number of





permutations of (n) quantities taken together, when there are p of the same sort, is $\frac{1 \cdot 2 \cdot 3 \cdot 4 \dots (n)}{1 \cdot 2 \cdot 3 \dots p}$;

And if there be (p) of one sort and q of another, the number of permutations is $\frac{1 \cdot 2 \cdot 3 \dots (n)}{(1 \cdot 2 \cdot 3 \dots p) \times (1 \cdot 2 \cdot 3 \dots q)}$

EXAMPLES

1. In how many different ways can the word "Trinities" be written?

The number of i's is 3,

The number of t's is 2,

$$n = 9$$

$$\text{Number required} = \frac{1 \cdot 2 \cdot 3 \dots 9}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3} = 30240.$$

2. In how many different ways can the word "Collegiate" be written?

There are two l's and two e's in the word.

$$n = 10$$

$$\text{The number sought} = \frac{1 \cdot 2 \cdot 3 \dots 10}{1 \cdot 2 \cdot 1 \cdot 2} = 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10.$$

If out of (n) quantities a, b, c , &c. it be required to find the number of permutations, when taken *two* together; take one of the quantities (a) apart from the rest, and place it successively before each of them; then there will be $\overline{n-1}$ permutations when (a) stands first; in like manner there will be $\overline{n-1}$ permutations when (b) stands first; the same is true of c, d , &c., each of the (n) quantities. Therefore all the permutations which these (n) quantities admit, taken *two together*, will be $n \cdot \overline{n-1}$.

In the above expression, suppose $n = m - 1$, then in a combination of $\overline{m-1}$ quantities, the number of permutations which can be formed out of them, taken two together, is $\overline{m-1} \cdot \overline{m-2}$. Now add another quantity (p) to the original number, and place it before each of the former permutations, and there will be $\overline{m-1} \cdot \overline{m-2}$ permutations, when (p) stands first; there will also be $\overline{m-1} \cdot \overline{m-2}$ permutations when each of the other quantities stands first: And as the whole number of quantities is now become (m), the number of permutations which can thus be formed out of (m) quantities, taken *three* together, is $m \cdot \overline{m-1} \cdot \overline{m-2}$. And in general the number of permutations of (m) things, taken (r) together, is $m \cdot \overline{m-1} \cdot \overline{m-2} \dots m - \overline{r-1}$.

The number of *Combinations* that can be formed out of (m) quantities, taken two together, is $m \cdot \frac{m-1}{2}$. For every combination of two quantities admits of 2 permutations $\therefore$ if the number of permutations be $m \cdot \overline{m-1}$, the number of combinations must be $m \cdot \frac{m-1}{2}$. By the same mode of reasoning, since every combination of three quantities admits of 3×2 permutations, it follows that if there be $m \cdot \overline{m-1} \cdot \overline{m-2}$ permutations of (m) quantities, taken three together, the number of combinations will be $\frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{2 \times 3}$. And in general, if they be taken (r) together, the number of combinations is

$$\frac{m \cdot \overline{m-1} \cdot \overline{m-2} \dots \overline{m-r+1}}{1 \cdot 2 \cdot 3 \cdot 4 \dots r}.$$





PROGRESSION.

THE quantities, a, b, c , are said to be in Arithmetical progression when $a-b : b-c :: a : a$.

The quantities a, b, c , are said to be in Geometrical Progression when $a-b : b-c :: a : b$.

The quantities a, b, c , are said to be in Harmonical Progression when $a-b : b-c :: a : c$.

ARITHMETICAL PROGRESSION.

SINCE $a-b$ is to $b-c$ in a ratio of equality, all the terms of an Arithmetical Series increase or decrease by a common difference. Thus the following series are in Arithmetical Progression.

1, 2, 3, 4, 5, 6, &c.

1, 3, 5, 7, 9, 11, 13, &c.

1, 5, 9, 13, 17, 21, &c.

12, 9, 6, 3, 0,

-1, -3, -5, -7, -9, -11, &c.

If then the first term of an arithmetical series be (a) and (b) be the difference between any two consecutive terms, the series will be $a, a+b, a+2b, a+3b, a+4b, a+5b, a+6b, \&c.$

It appears, from inspection, that the sum of any two terms equidistant from the first and last, will be the same

quantity in any series. Thus $a + a + 3b = \overline{a+b} + \overline{a+5b}$
 $= \overline{a+2b} + \overline{a+4b} = 2(a+3b)$, twice the middle term,
 when the series consists of an odd number of terms.
 Hence if we have to find the sum of any number of terms
 in a series, we may add the first and last terms together,
 and multiply this sum by half the number of terms. Thus
 to find the sum of ten terms of the series 2, 4, 6, 8, 10, 12,
 14, 16, 18, 20; $\overline{2+20} \times 5 (=110)$ gives the sum sought.

Suppose 100 stones be placed in a row, a yard from
 each other, and that a basket is placed in the same line,
 a yard from the first stone; how far will a person travel
 who starts at the basket, and brings each stone separately
 to it?

The spaces successively passed over are 2, 4, 6, 8 ...
 yards, continued to 100 terms; the first term of the series
 is 2, and the last is 200; $\therefore \overline{2+200} \times 50 = 10100$ yards,
 the space passed over.

As however the last term cannot always be conveniently
 found from the nature of the question, it is necessary to
 employ a general expression. Let a = first term, b =
 common difference, n = number of terms, S = sum of
 the series;

$$\text{Then } a + a+b + a+2b + a+3b + \dots + a+n-1.b = S$$

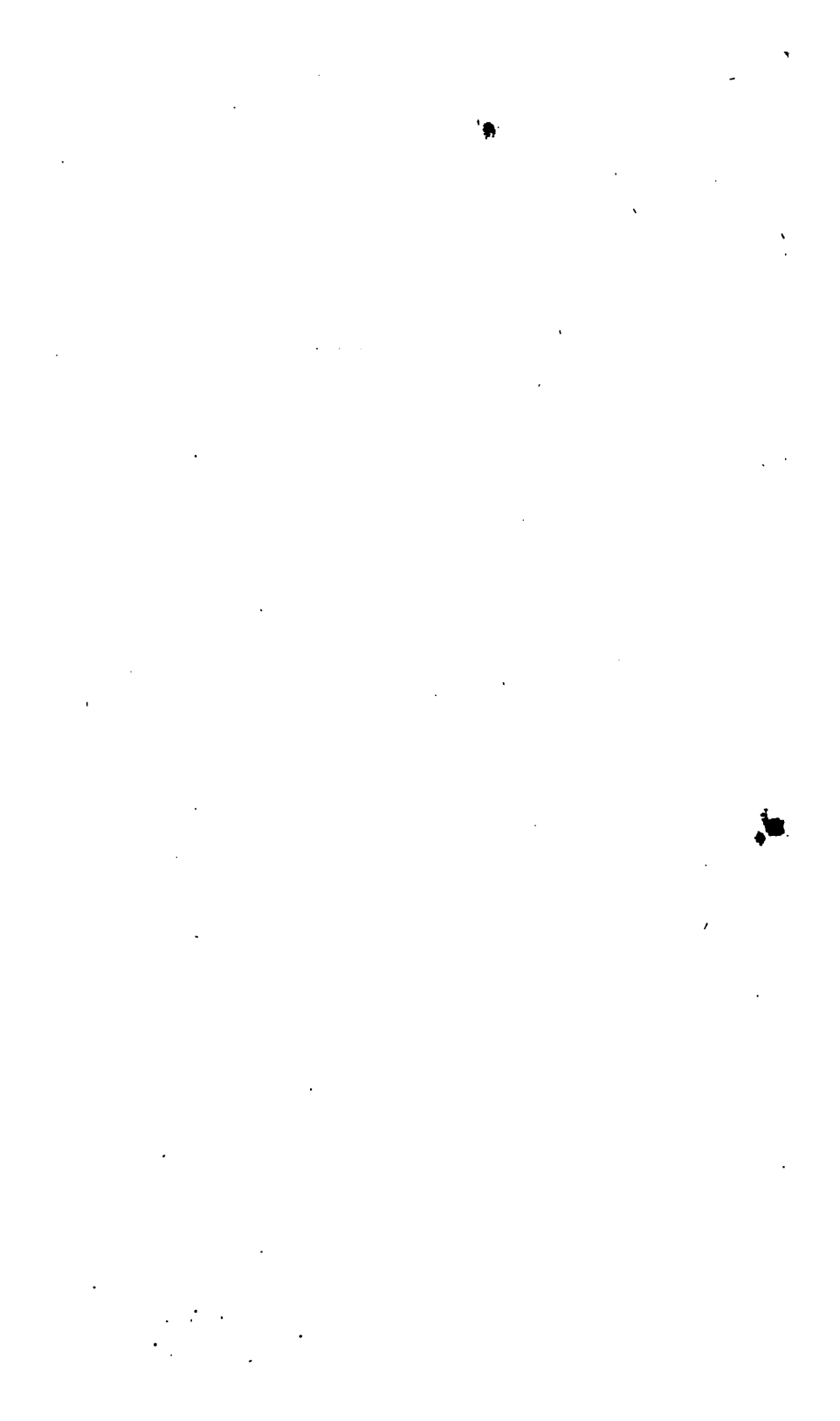
This equation must be true, if the terms be arranged
 in contrary order,

$$\text{Then } a+n-1.b + a+n-2.b + a+n-3.b \dots + a = S$$

$$\therefore 2a + n-1.b + 2a + n-1.b \dots \text{to } (n) \text{ terms} = 2S$$

$$\text{or } (2a + n-1.b) \times n = 2S$$

$$\therefore S = \overline{2a + n-1.b} \times \frac{n}{2}$$





Find the sum of the series $1 + 3 + 5 + 7 \dots$ continued to 101 terms.

$$a=1, b=2, n=101$$

$$\therefore S = \overline{2a+n-1} \cdot b \times \frac{n}{2}$$

$$= \overline{2+100} \times 2 \times \frac{101}{2}$$

$$= 202 \times \frac{101}{2} = 101 \times 101 = 10201.$$

Find the sum of a decreasing arithmetical series, of which the first term is 10, the common difference $\frac{1}{3}$, and the number of terms 21.

$$a=10, b=-\frac{1}{3}, n=21$$

$$S = \overline{20+20} \times -\frac{1}{3} \times \frac{21}{2}$$

$$= \overline{20-\frac{20}{3}} \times \frac{21}{2}$$

$$= \frac{40}{3} \times \frac{21}{2} = 20 \times 7 = 140.$$

The sum of a series is -460 ; the first term is 15; the number of terms is 20; find the common difference.

$$a=15$$

$$\overline{2a+n-1} \cdot b \times \frac{n}{2}$$

$$n=20$$

$$S=-460$$

$$-460 = \overline{30+19b} \times 10$$

$$\therefore -46 = 30 + 19b$$

$$\therefore 19b = -46 - 30 = -76$$

$$\therefore b = \frac{-76}{19} = -4.$$

The sum of an arithmetical series is 160; the first term is 4, and the number of terms is 10; find the common difference.

$$b = \frac{8}{3}.$$

Find the sum of 25 terms of an arithmetical series, of which the first term is $\frac{1}{2}$, and the common difference $\frac{1}{2}$.

$$S = 162\frac{1}{2}.$$

Find the n^{th} term, and the sum of (n) terms, of the series, 1, 3, 5, 7, 9, &c.

Find the n^{th} term, and the sum of (n) terms, of the series, 2, 4, 6, 8, &c.

* From the general expression find the value of (n) in terms of S , a and b .

$$n = \pm \sqrt{\frac{2S}{b} + \frac{(2a-b)^2}{2b}} - \frac{2a-b}{2b}.$$

Insert 6 arithmetic means between 5 and 11.

Let x = common difference.

Then the series is 5, $5+x$, $5+2x$, $5+3x$, $5+4x$, $5+5x$, $5+6x+11$.

$$\text{Then } 5+11 = 5+x+5+6x = 10+7x$$

$$\therefore 7x=6 \quad \therefore x=\frac{6}{7},$$

And the series is 5, $5\frac{6}{7}$, $6\frac{2}{7}$, $7\frac{4}{7}$, $8\frac{2}{7}$, $9\frac{2}{7}$, $10\frac{4}{7}$, 11.

Insert (n) arithmetic means between A and B .

Let x = common difference.

The series is $A, A+x, A+2x, \dots, A+\overline{n-1} \cdot x, A+nx, B$.

$$\therefore A+B = A+x+A+nx$$

$$\therefore B-A = \overline{n+1} \cdot x$$

$$\therefore x = \frac{B-A}{n+1}.$$

Insert 7 arithmetic means between 4 and 9.

Find the sum of the following series ; $\overline{a+y}^2, a^2+y^2, \overline{a-y}^2$ &c.

* vide page 64, Part II, for the work

$$\text{the } n^{\text{th}} \text{ term} = 2n - 1, \quad S = n^2 - n.$$

$$\text{the } n^{\text{th}} \text{ term} = 2n \text{ ~~mp~~.} \quad S = n^2 + n.$$



GEOMETRICAL PROGRESSION.

SINCE when $a-b : b-c :: a : b$, a, b, c are said to be in Geometrical Progression, we have, by multiplying means and extremes, $ab - b^2 = ab - ac \therefore ac = b^2$, or $a : b :: b : c$, or $\frac{a}{b} = \frac{b}{c}$.

It appears then, that in a geometrical series, every term is in a given ratio to that which precedes it. Thus the following series are in Geometrical Progression :

1, 2, 4, 8, 16, &c.

1, 3, 9, 27, 81, &c.

1, 4, 16, 64, 256, &c.

1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c.

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$, &c.

Investigate the general expression for finding the sum of a geometrical series.

Let (a) be the first term, (r) the common ratio, (n) the number of terms, (S) the sum of the series ; then the series

$$\text{is } a + ar + ar^2 + ar^3 \dots ar^{n-1} = S$$

$\therefore$ multiplying by r , $ar + ar^2 + ar^3 + \dots ar^n = rS$

$$\therefore ar^n - a = rS - S = S \cdot r - 1$$

$$\therefore S = \frac{ar^n - a}{r - 1}$$

K

EXAMPLES.

Find the sum of 9 terms of the series 1, 2, 4, 8, &c.

$$a=1, n=9, r=2$$

$$S = \frac{ar^n - a}{r - 1} = \frac{1 \times 2^9 - 1}{2 - 1} = 512 - 1 = 511.$$

Find the sum of 10 terms of the series 9, 27, 81, &c.

$$a=9, r=3, n=10$$

$$S = \frac{ar^n - a}{r - 1} = \frac{9 \times 3^{10} - 9}{3 - 1} = \frac{9 \times (3^{10} - 1)}{2} = \frac{9}{2} \times (3^{10} - 1) = 265716.$$

Find 6 terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c.

$$a=1, r=\frac{1}{3}, n=6$$

$$S = \frac{ar^n - 1}{r - 1} = \frac{1 \times \frac{1}{3^6} - 1}{\frac{1}{3} - 1} = \frac{\frac{1}{3^6} - 1}{-\frac{2}{3}} = \frac{1 - 1 \times \frac{1}{3^6}}{\frac{2}{3}} = \frac{3^6 - 1}{\frac{2}{3} \times 3^6} = \frac{3^6 - 1}{2 \times 3^5} = \frac{728}{486} = \frac{364}{243}.$$

The value of (r) is always obtained from dividing any term by the one immediately preceding it.

If r be a proper fraction, and the terms continued in infinitum, each succeeding term will be less than the preceding one, and the latter terms of the series will be indefinitely small. To find (S) in this case, we have

$$\begin{array}{rcl} a + ar + ar^2 + ar^3 + \&c. & \dots = S \\ ar + ar^2 + ar^3 + \&c. & = rS \end{array}$$

$$\therefore a = S - rS = S \cdot \overline{1 - r} \therefore S = \frac{a}{1 - r}$$



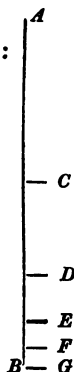
Sum the series $1 + \frac{1}{2} + \frac{1}{4}$, &c. in infinitum.

$$a=1, r=\frac{1}{2}$$

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

This may be illustrated in the following manner :

Suppose AB to be a line 2 feet long, bisected in C . Again, let CB be bisected in D ; DB again bisected in E ; EB bisected in F : then we have $AC + CD + DE + EF + FG + \&c.$ representing $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$ and by continuing the subdivisions it is clear that we can approach nearer than by any assignable difference to the whole line AB , which is 2 feet in length.



From observing the general series $a, ar, ar^2, ar^3, ar^4 \dots ar^{n-1}, ar^n$, it appears that the product of any two terms equidistant from the two ends of the series is always the same; thus $a \times ar^n = ar \times ar^{n-1} = ar^2 \times ar^{n-2}$, &c.

Find the sum of $1 + \frac{1}{3} + \frac{1}{9} + \&c.$ in infinitum.

$$a=1, r=\frac{1}{3}$$

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{3-1} = \frac{3}{2}$$

Find the sum of $2 + \frac{2}{3} + \frac{2}{9} + \&c.$ in infinitum.

$$S = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{6}{3-1} = 3$$

Find the sum of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \&c.$

$$r = -\frac{1}{3}$$

$$S = \frac{a}{1-r} = \frac{1}{1+\frac{1}{3}} = \frac{3}{3+1} = \frac{3}{4}$$

Sum $3 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \&c.$ in infinitum.

$$r = -\frac{1}{3}$$

$$S = \frac{a}{1-r} = \frac{3}{1+\frac{1}{3}} = \frac{27}{10} = 2\frac{7}{10}.$$

Find the sum of $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \&c.$ in infinitum.

$$a = \frac{3}{10}, r = \frac{1}{10}$$

$$S = \frac{a}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{3}{9} \text{ or } \frac{1}{3}.$$

Find the sum of $\frac{12}{10^2} + \frac{12}{10^4} + \frac{12}{10^6} + \&c.$ in infinitum.

$$a = \frac{12}{10^2}, r = \frac{1}{10^2}$$

$$S = \frac{a}{1-r} = \frac{\frac{12}{10^2}}{1-\frac{1}{10^2}} = \frac{12}{99}.$$

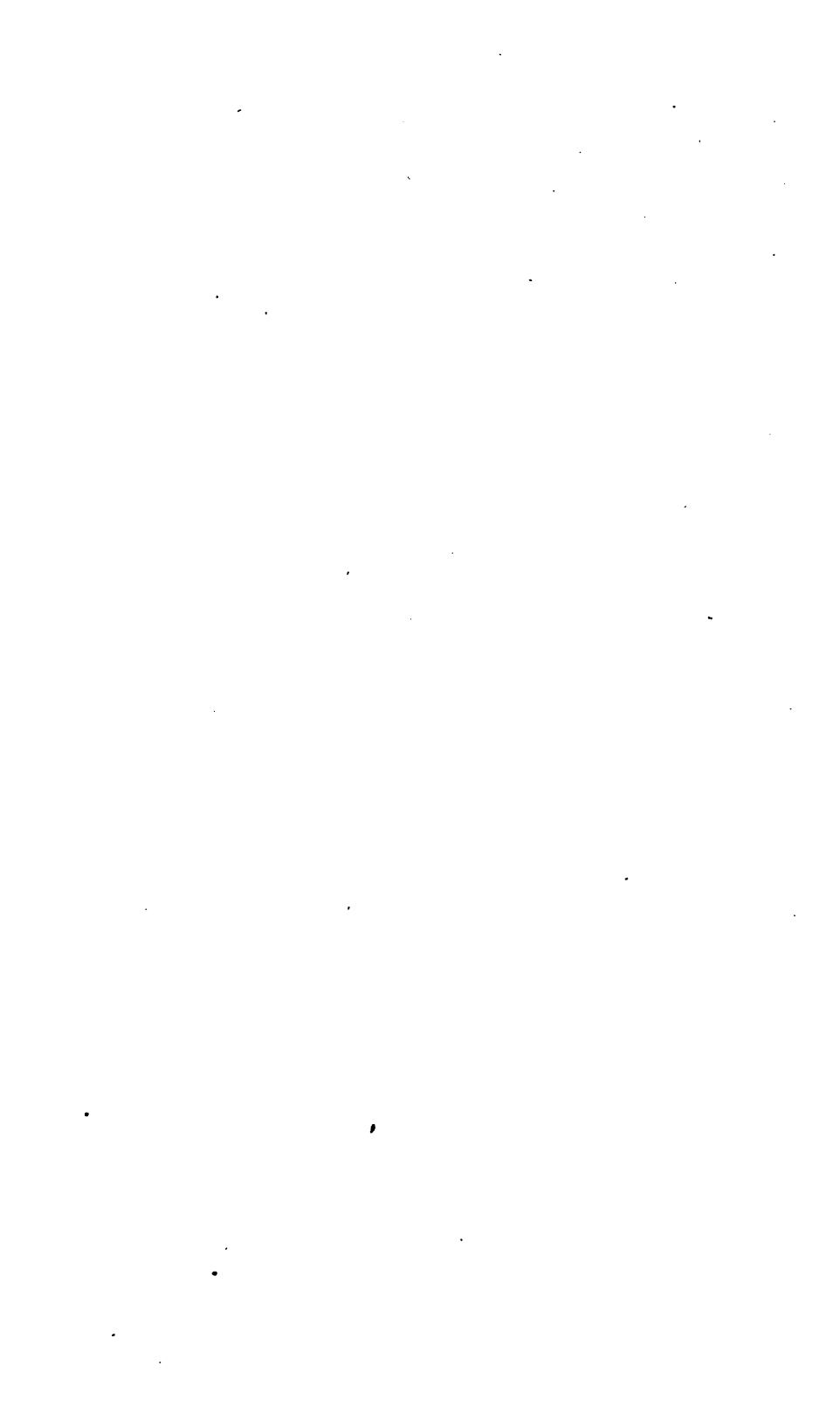
The series last proposed is evidently the same as the recurring decimal $\cdot 121212, \&c.$

In like manner if it be required to find the value of $123123123, \&c.$; this number is the same as

$$\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} + \frac{3}{10^6} + \&c.; \text{ or as}$$

$$\frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \&c.$$

$$S = \frac{a}{1-r} = \frac{\frac{123}{10^3}}{1-\frac{1}{10^3}} = \frac{123}{999}.$$





In like manner the amount of any recurring decimal, as $\cdot 714285714285$, &c., is $\frac{714285}{999999}$.

A simpler, though not essentially different, method of finding the value of a recurring decimal is this:

Take $\cdot 123123123$, &c., for instance;

Let $\cdot 123123123$, &c. = S

$\therefore 123 \cdot 123123$ &c. = $1000S$

$\therefore 123 = 1000S - S = 999S$

$\therefore S = \frac{123}{999}$.

Find the value of $\cdot 41666$, &c.

Or $\frac{4}{10} + \frac{1}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \frac{6}{10^5}$

First, $\frac{4}{10} + \frac{1}{10^2} = \frac{41}{10^2}$; to find the rest of the series, we have $a = \frac{6}{10^3}$, $r = \frac{1}{10}$

$$S = \frac{a}{1-r} = \frac{\frac{6}{10^3}}{1 - \frac{1}{10}} = \frac{6}{900}$$

$$\therefore \frac{41}{100} + \frac{6}{900} = \frac{375}{900} = \frac{5}{12}$$

Insert 3 geometrical means between 2 and 32.

Let the series be 2, $2r$, $2r^2$, $2r^3$, 32

$\therefore 2 \times 32 = 2r \times 2r^3 = 4r^4$

$\therefore 16 = r^4$, $\therefore r = 2$

And the series is, 2, 4, 8, 16, 32.

To insert (n) geometrical means between A and B .
Let (r) be the common ratio.

Let the series be $A, Ar, Ar^2, \dots, Ar^n, B$

$$\therefore AB = Ar^{n+1} \therefore r^{n+1} = \frac{B}{A}$$

$$r = \sqrt[n+1]{\frac{B}{A}}$$

Insert 4 geometric means between 5 and 16.

Insert 5 geometric means between 3 and 17.

Insert 6 geometric means between 4 and 18.

Insert 7 geometric means between 8 and 28.

Insert 8 geometric means between 9 and 30.



LOGARITHMS

ARE numbers invented to facilitate Arithmetical calculations; and are so contrived that the Addition and Subtraction of Logarithms may supersede the labour of Multiplication and Division of the natural numbers.

If we assume a certain quantity (a) with a variable index (x), it is evident that, by taking every possible value of (x), the quantity a^x may represent all numbers whatever. If then a^x denote a certain number, (x) is called the Logarithm of that number, and (a) is called the base of the system.

Suppose then that (a) be made equal to 10, and x be successively = 1, 2, 3, &c. $10^0, 10^1, 10^2, 10^3, 10^4$, &c., or 1, 10, 100, 1000 are numbers whose logarithms are 0, 1, 2, 3, 4, &c.

It is manifest that the numbers 1, 10, 100, 1000, &c. are in Geometric Progression, and that their Logarithms, 0, 1, 2, 3, &c. are an Arithmetical series. If then we insert Geometric means between two of the numbers, and the same number of Arithmetic means between their Logarithms, we may, by continuing that process, find Logarithms for all numbers whatever. Thus the Logarithm of 10 is 1; the Logarithm of 100 is 2; now the Geometric mean between 10 and 100 is $10\sqrt{10}$ or 31.622777. The Arithmetic mean between 1 and 2 is 1.5; therefore 1.5 is the Logarithm of 31.622777.

Again the Logarithm of 1 is 0

And the Logarithm of 10 is 1

Now the Geometric mean between 1 and 10 is $\sqrt{1 \times 10}$ or 3·1622777.

And the Arithmetic mean between 0 and 1 is ·5,

Therefore the Logarithm of 3·1622777 is ·5.

Again, if we find the Geometric mean between 3·1622777 and 10; and the Arithmetic mean between ·5 and 1; we obtain another number and its Logarithm. But this process is evidently so tedious, that a complete Table of Logarithms could not be completed within a period of several years. Neither would the labour be much abridged if we had, in every case, to find by trial the value of (x) in the equation $10^x = N$, where N is taken successively to represent every possible number. Suppose we want to find the Logarithm of 5, we have $10^x = 5$; if we make $x = \frac{1}{2}$, we find $10^{\frac{1}{2}}$, or $\sqrt{10}$, to be too little; if we assume $x = \frac{3}{4}$, we have $10^{\frac{3}{4}}$ or $\sqrt[4]{1000}$ to be too great; if we next make $x = \frac{2}{3}$, we have $10^{\frac{2}{3}}$, or $\sqrt[3]{100}$, = 4·64, which is less than the proposed number, 5; that is, $\frac{2}{3}$, or 6666, &c. is less than the logarithm sought. On further trial we discover that $10^{·698700} = 5$; thus we at length find that ·698700 is the log. of the number 5, in a system whose base is 10. Again, to find the log. of 2, we have $10^x = 2$; by trial, we find that $10^{·3010300} = 2$, or ·3010300 is the log. of 2. In like manner we find that $10^{·477123} = 3$, or ·477123 is the log. of 3. By thus substituting every possible value for x , 10^x may represent any number whatever.

The log. of the product of two numbers is the sum of their logarithms.

Let one number (A) be represented by a^x ; $\therefore x = \text{log. of } A$

Let the other number (B)..... by a^y ; $\therefore y = \text{log. of } B$

Then $A \times B = a^x \times a^y = a^{x+y}$; $\therefore (x + y)$ is log. of $A \times B$.





In like manner $\frac{A}{B} = a^{x-y}$; $\therefore (x-y)$ is log. of $\frac{A}{B}$, or the log. of the quotient of two numbers is the difference of their logarithms.

The log. of A^m is (m) times the log. of A .

Let $A = a^x$, where (x) is log. of A , then $A^m = a^{mx}$; but (mx) is the log. of a^{mx} or of A^m , and $\therefore$ is m times the log. of A .

And if (m) be assumed equal $\frac{1}{n}$; the log. of $A^{\frac{1}{n}}$ is $\frac{1}{n}$ log. of A .

The base (a) of the system may be assumed of any value; but there is a decided advantage in making it = 10; for supposing we have found

the log. of 3526 to be 3.5472823; then
since the log. of 10 is 1

the log. of $\frac{3526}{10}$ or of 352.6 is 2.5472823

the log. of $\frac{3526}{10^2}$ or of 35.26 is 1.5472823

the log. of $\frac{3526}{10^3}$ or of 3.526 is .5472823

the log. of .3526 is $\bar{1}$.5472823

the log. of .03526 is $\bar{2}$.5472823

Thus, by moving the decimal point in the number, and changing the integral part of the Logarithm, the same figures serve to express several numbers and their Logarithms.

Since $a^x = N$; if $x=0$, a^0 or $N = 1$

therefore in every system, the log. of 1 = 0.

If $x = -1$, in the common system where $a = 10$

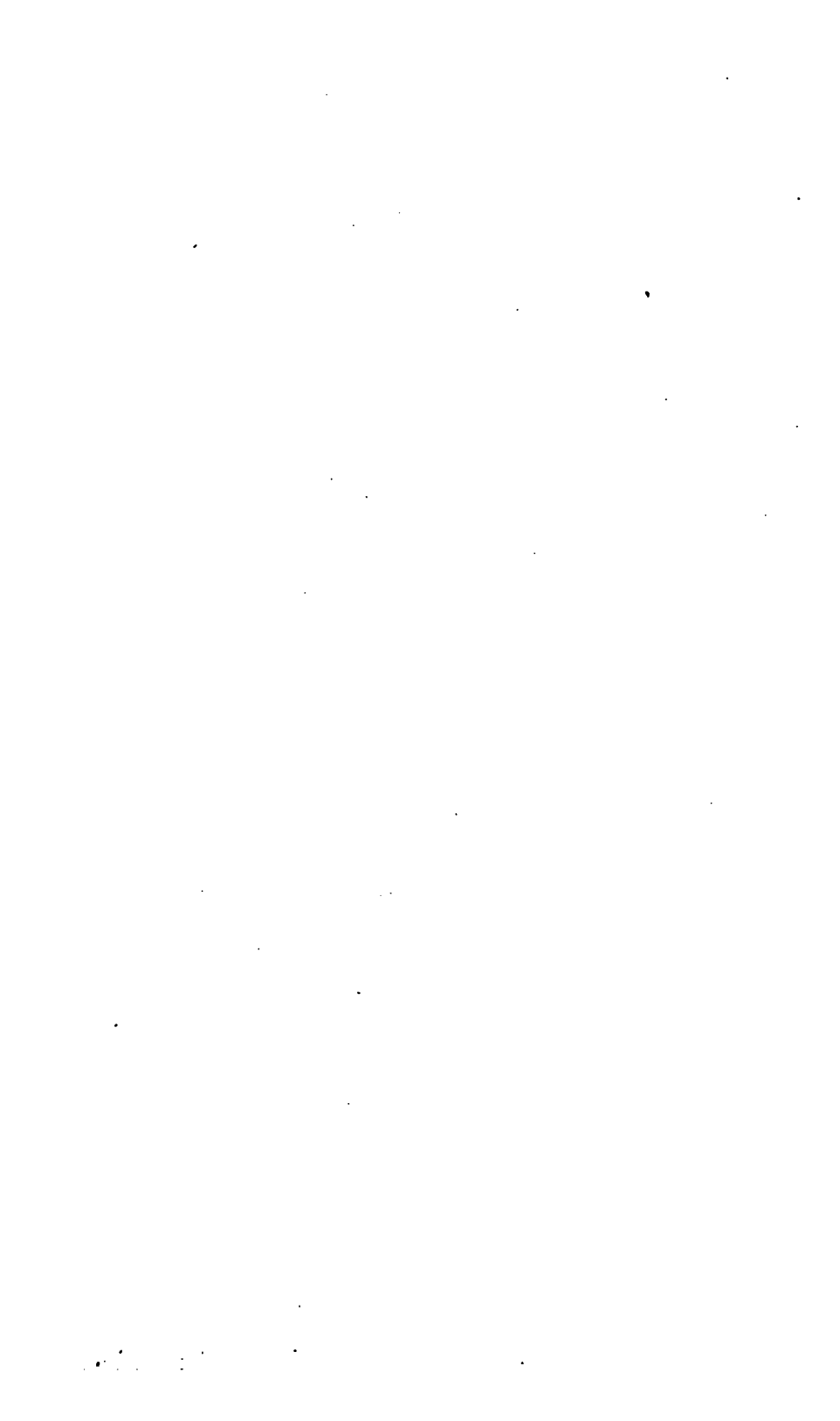
$$10^{-1} = \frac{1}{10} = \cdot 10000; \text{ whose log. is } -1$$

$$10^{-2} = \frac{1}{10^2} = \cdot 01000; \text{ whose log. is } \bar{2}$$

$$10^{-3} = \frac{1}{10^3} = \cdot 00100; \text{ whose log. is } \bar{3}.$$

Thus the log. of a proper fraction is always negative.

If (e) be the base of one system, and (a) the base of another, and a given number $(N) = e^x = a^y$, . . . $e = a^{\frac{x}{y}}$.



I N T E R E S T .

Definition.

Simple Interest is an allowance made for the loan of money, according to a certain yearly rate.

When the interest is left unpaid, after it becomes due, it is annexed to the original loan, and this joint sum becomes the *principal*; and the interest accruing from it is called *Compound Interest*.

SIMPLE INTEREST.

If (r) be the interest of £1 for 1 year, $2r$ is the interest of £1 for 2 years; and (nr) is the interest of £1 for (n) years: consequently pnr is the interest of (p) £ for (n) years; and $p + pnr$ is the *amount* of p £ for (n) years.

EXAMPLES.

What is the amount of £547 : 15s. : 0d. for 3 years, at 5 per cent. per annum, simple interest.

$$p = 547\frac{3}{4} = 547.75$$

$$n = 3; \quad r = \frac{5}{100} = .05$$

$$nr = .15$$

$$\therefore prn = £82 : 3s. : 3d. = \text{interest.}$$

$$\text{The amount} = \begin{cases} 547 : 15 : 0 \\ 82 : 3 : 3 \end{cases}$$

$$£629 : 18 : 3$$

2. What is the amount of £690 for 3 years, at $4\frac{1}{2}$ per cent per annum, simple interest?

$$p = 690$$

$$r = \cdot 045$$

$$n = 3$$

$$100 : 1 :: 4\frac{1}{2} : r$$

$$\text{or } 100 : 1 :: 4\cdot 5 : r = \frac{4\cdot 5}{100} = \cdot 045$$

$$\therefore prn = 3 \times 690 \times \cdot 045$$

$$= 93\cdot 150$$

$$\text{£ } s. \text{ } d.$$

$$= 93 : 3 : 0$$

$$\therefore \text{Amount } (=p + prn) = 783 : 3 : 0.$$

3. What is the interest of £205 : 15s. : 0d. for $\frac{1}{4}$ of a year, at 4 per cent per annum?

$$p = 205\cdot 75$$

$$r = \cdot 04$$

$$n = \cdot 25$$

$$\therefore prn = 2\cdot 0575$$

$$20$$

$$1\cdot 1500$$

$$12$$

$$1\cdot 8000$$

$$4$$

$$\text{Interest} = \text{£}2 : 1s. : 1\frac{3}{4}d. \quad 3\cdot 2$$

4. What is the interest of £47 : 10s. : 0d. for 4 years and 52 days at $4\frac{1}{2}$ per cent?

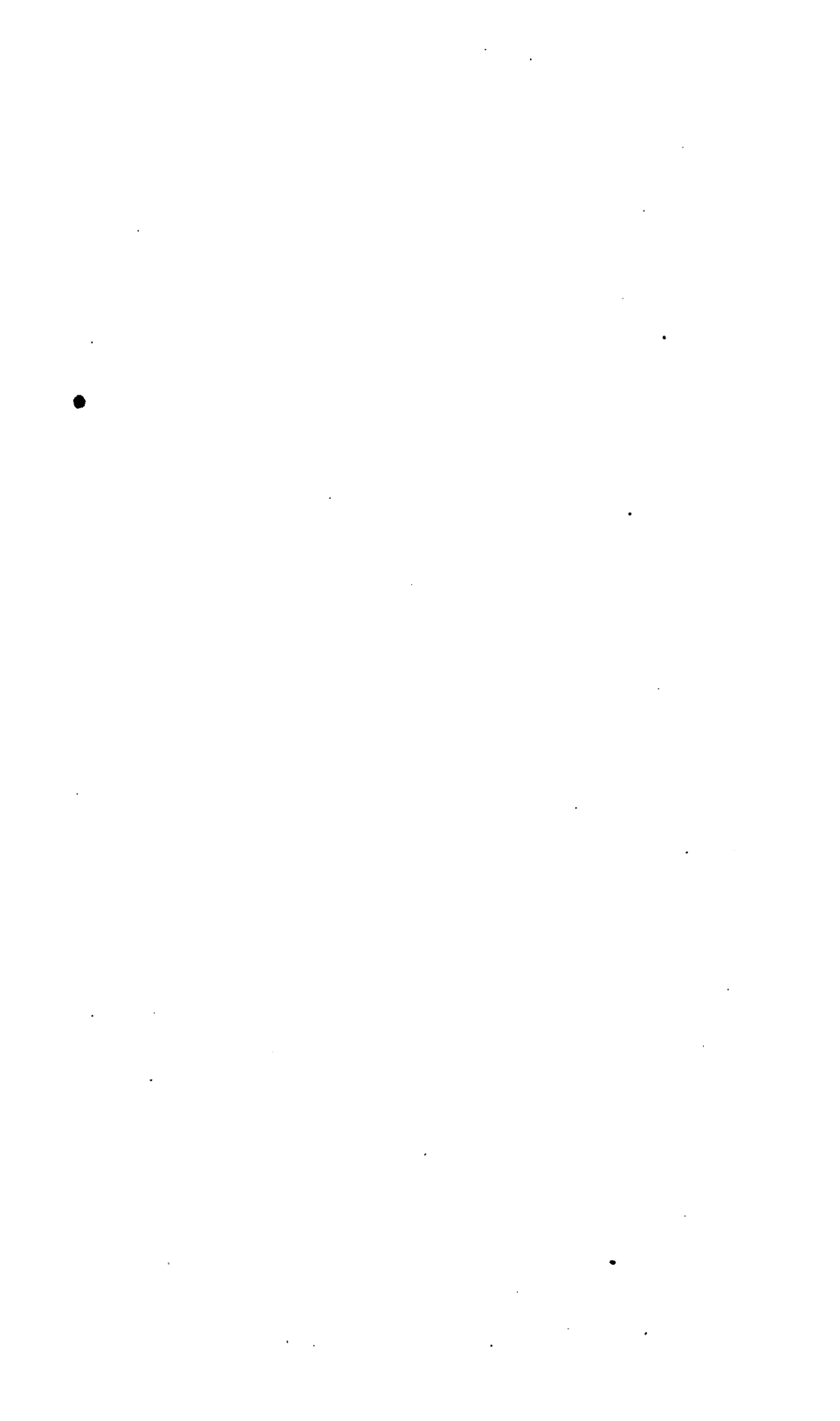
$$p = 47\cdot 5$$

$$r = \cdot 045$$

$$n = 4\frac{52}{365} = 4\cdot 14247 \quad \text{£ } s. \text{ } d.$$

$$prn = 22641115 \times 47\cdot 5 = 8 : 17 : 1$$

$$r = \frac{4\frac{1}{2}}{100} = \frac{4.5}{100} = .045$$



Since the amount $(M) = P + Prn = P \times (1 + rn)$
 $\therefore P$, the present value of a sum of money (M) due (n)
 years hence, is $= \frac{M}{1 + rn}$.

Find the present value of £500 due 2 years hence, at 5 per cent. per annum, simple interest.

$$M = 500, r = .05, n = 2$$

$$\therefore \text{Present value } (P) = \frac{M}{1 + rn} = \frac{500}{1.1} = £454 : 10 : 10\frac{3}{4}.$$

Discount is the difference between the sum due at a future period and its present value. The *discount* in the last example is the difference between £500, and £454 : 10s. : 10 $\frac{3}{4}$ d. : which is £45 : 9s. : 1 $\frac{1}{4}$ d.

A person bought goods for £150 ready money, and sold them again for a £200 bill, payable $\frac{3}{4}$ of a year afterwards; what was his gain, supposing discount at 5 per cent.?

The present value of £200, payable $\frac{3}{4}$ of a year hence, is

$\frac{P}{1 + nr} = \frac{200}{1.0375} =$	$\begin{array}{r} \text{£} \quad \text{s.} \quad \text{d.} \\ 192 : 15 : 5 \end{array}$	$\begin{array}{l} n = .75 \\ r = .05 \end{array}$
	$\begin{array}{r} \text{From } 192 : 15 : 5 \\ \text{Deduct } 150 : 0 : 0 \\ \hline \end{array}$	$\begin{array}{l} \hline nr = .0375 \end{array}$
$\text{his profit is } \underline{\underline{£ 42 : 15 : 5}}$		

This is the proper mode of calculating discount. The usual mode with bankers and brokers is to compute interest upon the sum due at a future time, and to deduct that in-

terest from the sum. Thus if a banker give the present value of £100 due a year hence, at 5 per cent. discount, he will pay but £95 0s. 0d.; but if this £95 be immediately put out to interest, it will bear an interest of only 95 shillings in one year: there is therefore a loss of nearly $\frac{1}{4}$ per cent. to the dealer, and of gain to the banker. The sum he ought to receive is $\frac{P}{1+nr} = \frac{100}{1.05} = \text{£}95 \text{ 4s. } 9\frac{1}{4}\%$ for this sum will, in the course of one year, at 5 per cent., amount to £100.

$$\begin{array}{rcl}
 \text{Thus } p = 95 : 4 : 9 = 95.2375 & & \text{£.} \\
 r = & = & .05 \\
 n = 1 & & \text{£. s. d.} \\
 \therefore prn = 4.761875 = 4 : 15 : 2\frac{1}{4} & & \\
 p = 95 : 4 : 9\frac{1}{4} & & \\
 \hline
 \therefore p + prn = \text{£}100 : 0 : 0 & &
 \end{array}$$

In the question last proposed a broker would calculate the interest of £200 for $\frac{3}{4}$ of a year, and deduct this sum from £200.

$$\begin{array}{rcl}
 \text{Thus } \text{£}200 : 0\text{s.} : 0\text{d.} \\
 \text{Interest} = 7 : 10 : 0
 \end{array}$$

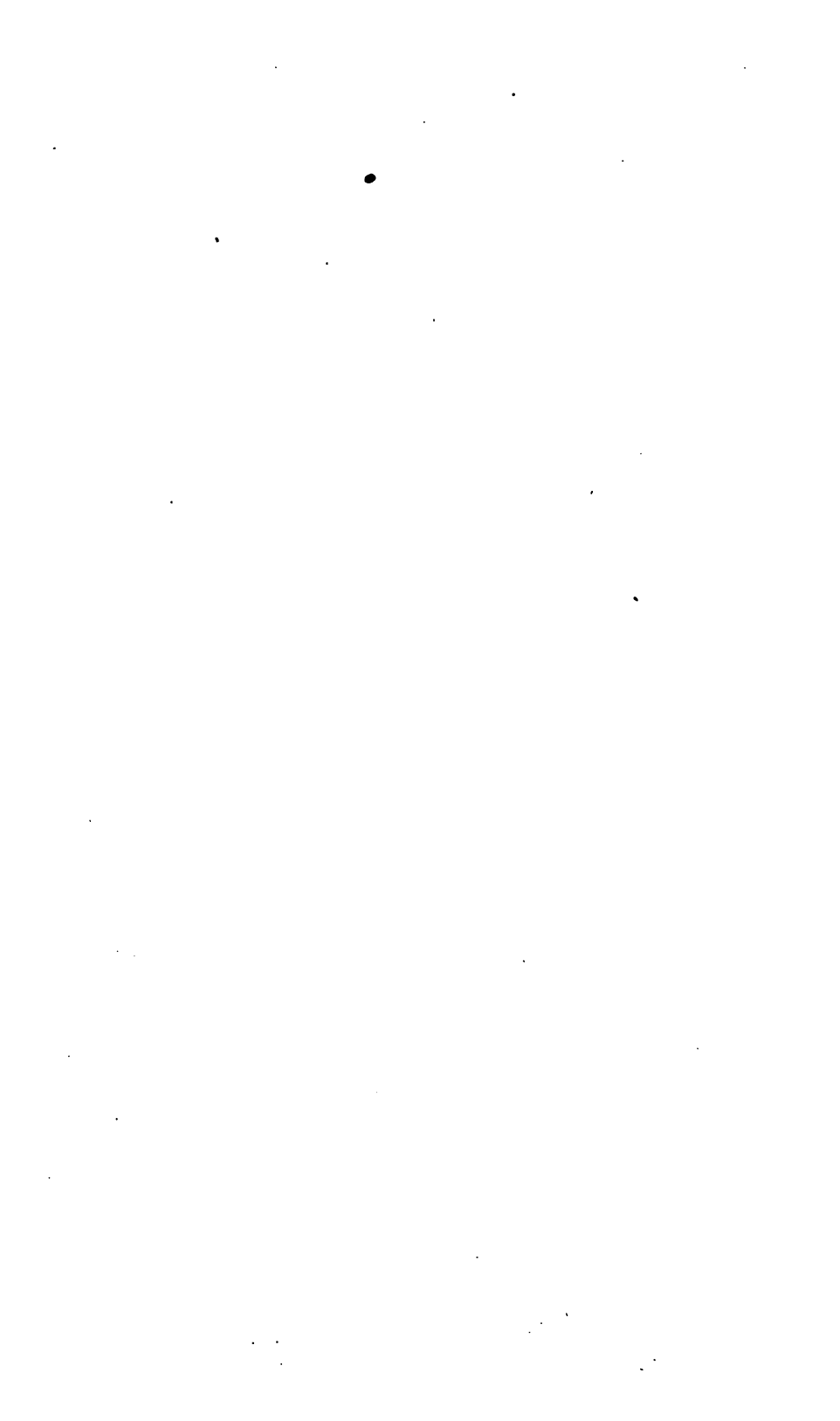
$\text{£}192 : 10 : 0$; which differs from the proper sum by 5s. : 5d.

On the 4th July a merchant draws a bill for £85 10s. 0d. due September 8th, what is the proper discount on it, at the rate of 5 per cent. per annum?

$$\text{The bill is drawn at 69 days, } \therefore n = \frac{69}{365} = .1809$$

$$\begin{array}{rcl}
 \text{The present value is } \frac{P}{1+nr} = \frac{85.5}{1.009045} = \text{£}84 \text{ 14s. } 8\frac{1}{4}\text{d.} \\
 \therefore \text{the discount will be } 15\text{s. } 3\frac{1}{2}\text{d.}
 \end{array}$$





A mode of computation equally erroneous is adopted by merchants in finding *Equation of Payments*.

Thus, if *A* has to pay £50 in 6 months, and £80 in 10 months, the equated time for paying the whole sum is usually calculated on the following principle :

Let x = the time sought,

Then the interest upon £50 for $(x-6)$ months $= 50 \times r \times (x-6)$

The interest upon £80 for $(10-x)$ months $= 80 \times r \times (10-x)$

If those sums be made equal to each other,

$$50rx - 50 \times 6r = 800r - 80rx$$

$$\therefore 50x + 80x = 800 + 300$$

$$\text{or } x = \frac{1100}{130} = \frac{110}{13} = 8\frac{6}{13} \text{ months.}$$

And in general, if $(P\pounds)$ be due (T) months hence, and $(p\pounds)$ be due (t) months hence, the equated time, by this method, for paying the whole is $\frac{TP+tp}{P+p}$.

Suppose £100 to be due one year hence, and £105 to be due 3 years hence, find the equated time for paying both sums, without profit or loss, supposing simple interest at 5 per cent.

By the last method the equated time is

$$\frac{100 \times 1 + 105 \times 3}{100 + 105} = \frac{415}{205} = 2\frac{5}{41} = 2\frac{1}{8}.$$

But that the true equated time is precisely 2 years, is manifest from the nature of the question: for £100 delayed for one year amounts to £105; and £105, paid with discount, a year before it is due, is £100. Hence the two sums to be paid 2 years hence would be £105 and £100; and these are the two sums stated in the question. The following investigation gives the true equated time :

If P and p be two sums of money due at the expiration at N and n years respectively; the equated time (x) for paying the two sums together is determined by solving this

$$\text{equation; } (N+n+\frac{P+p}{pr})x-x^2=\frac{PN+pn}{pr}+Nn.$$

If $P\pounds$ (due N years hence) be paid (x) years before it is due, its value at the time of payment is $\frac{P}{1+r(N-x)}$, and the amount of $p\pounds$ in $(x-n)$ years is $p+pr(x-n)$;

These two sums then, payable at the time (x) must be made equal to the two debts $P\pounds$ and $p\pounds$.

$$\therefore \frac{P}{1+Nr-xr} + p + prx - prn = P + p$$

$$\therefore prx + pr^2Nx - pr^2x^2 - prn - Nnpr^2 + pr^2nx + P = P + p + NPr - Prx$$

divide both sides by pr^2

$$\therefore x\left(\frac{1}{r} + N + n + \frac{P}{pr}\right) - x^2 = \frac{NP}{pr} + Nn + \frac{n}{r}$$

$$\therefore x\left(N+n+\frac{P+p}{pr}\right) - x^2 = \frac{NP+pn}{pr} + Nn.$$

Apply this expression to the solution of the last question, $p=100$, $P=105$, $n=1$, $N=3$, $r=.05$; then the expression becomes

$$x\left(3+1+\frac{105+100}{5}\right) - x^2 = \frac{3 \times 105 + 100 \times 1}{5} + 3 \times 1$$

$$\text{or } x(4+41) - x^2 = 83 + 3$$

or $x^2 - 45x = -86$; the conditions of which equation are answered when 2 is substituted for x .

Suppose $\pounds 400$ to be payable in 2 years; and $\pounds 2100$ in 8 years; find the equated time when both sums may be paid, reckoning 5 per cent. interest.

This expression may also be investigated thus.

A owes £ P due at the end of N years
 & £ p ----- n -----
 from the present time. To find the period at
 which he should pay both sums without gain or
 Let x be the time.

then $(x-n)$ is the extra time ^{during} which A has the use
 of money; & $\therefore p(x-n)r$ is the interest.

Again $(N-x)$ is the time which he pays £ P before
 it is due.

$$\therefore \text{Discount} = P - \frac{P}{1 + (N-x)r} = \frac{P(N-x)r}{1 + (N-x)r}$$

which is his loss, & must $\therefore$ = his gain

$$\therefore p(x-n) = \frac{P(N-x)r}{1 + (N-x)r}$$

hence $px - pn + Nprx - Npnr + pnrx - px^2r = Pn$
 hence by transposing & dividing by $-pr$

$$x^2 - x(N+n + \frac{P+p}{pr}) = - \left\{ \frac{PN + pn}{pr} + Nn \right\}$$

which is the same equation.

The error in the common method, is by calculating
 the loss, not by discount but interest, & making
 it $\therefore$ = $P(N-x)r$

In which case

$$p(x-n) = P(N-x)r$$

$$\therefore x = \frac{PN + pn}{P + p}$$



Here $P=2100$; $p=400$; $N=8$, $n=2$, $r=.05$. The expression becomes

$$x \left(8 + 2 + \frac{2100+400}{20} \right) - x^2 = \frac{16800+800}{20} + 16$$

$\therefore 135x - x^2 = 896$; from which equation we find $x=7$.

If we employ the common method we have the time required $= \frac{2100 \times 8 + 400 \times 2}{2100 + 400} = 7\frac{1}{25}$, an answer which gives more than a fortnight too much.

If three sums, P , p , and p' , be due at three different periods, the equated time at which they may all be paid together, is known by first finding the time (t) at which P and p may be paid together; and then finding the time at which $(P+p)$ £ due (t) months hence, and p' £ due (n) months hence may be paid together.

A debt of £1000 is paid by instalments in the following manner; half of it is paid at the end of 8 months, one-third of the sum at the end of a year, and the remainder at the expiration of a year and a half. It is required to find the equated time when the whole may be paid without gain or loss.

ON NOTATION, NUMBERS, &c.

THE *local value* in any system of Notation is the number by which a digit is multiplied in changing its place from the right hand to the left. And in every system the local value expresses the number of digits employed in that system. In the common, or denary, scale of notation, the local value is 10, and 10 is the number of digits employed. Any number, as 36754, may be written thus: $(3 \times 10^4) + (6 \times 10^3) + (7 \times 10^2) + (5 \times 10^1) + (4 \times 10^0)$. And, in general, if $a, b, c, d \dots e, g$, &c., be the digits of a number, in a system whose local value is (x) , the number itself is $gx^m + ex^{m-1} \dots + dx^3 + cx^2 + bx + a$.

Since, in the common system of notation, the value of a digit decreases in a tenfold proportion, as it changes its place from left to right, it is manifest that all terms on the right of the unit digit are *fractions*, and as their denominators are all powers of 10, they are called *Decimal Fractions*. The integral and fractional parts are separated by a point. Thus 463.75 is the same as $4 \times 10^2 + 6 \times 10^1$

$+ 3 \times 10^0 + \frac{7}{10} + \frac{5}{10^2}$. Any decimal fraction then, as .8765

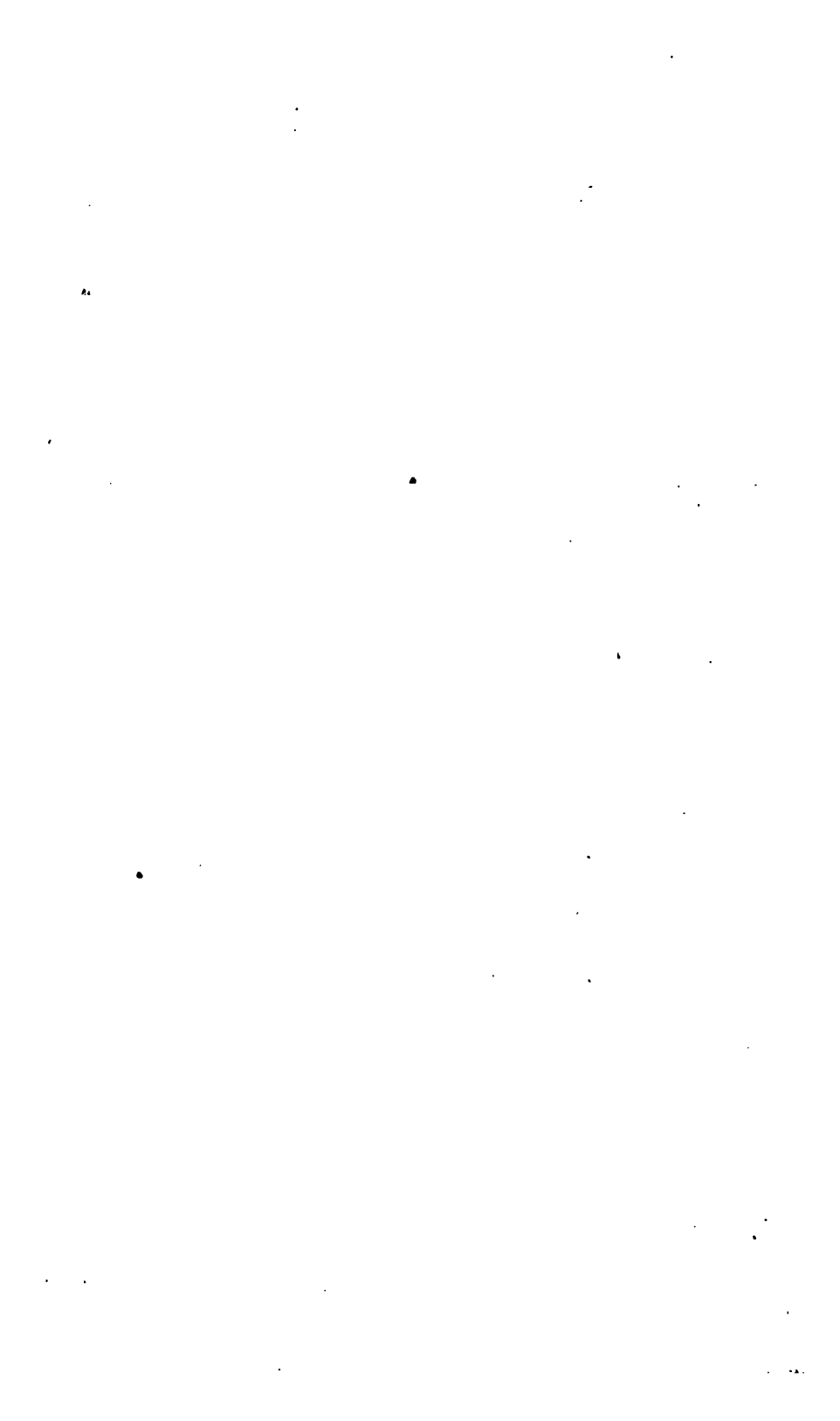
is the same as $\frac{8765}{10^4}$; for it is the same as $\frac{8}{10} + \frac{7}{10^2}$

$+ \frac{6}{10^3} + \frac{5}{10^4}$ which is $= \frac{80000 + 700 + 60 + 5}{10^4} = \frac{8765}{10^4}$.

The number 4.0372 is the same as $\frac{40372}{10^4}$, for it is the

$$F \approx 4 \times 10^2 + 6 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1} + 5 \times 10^{-2}$$

$$F \approx 21.0765 \times 10^{-5}$$



same as $4 + \frac{3}{10^1} + \frac{7}{10^2} + \frac{2}{10^3}$; which is =

$$\frac{40000 + 300 + 70 + 2}{10^4} = \frac{40372}{10^4}; \text{ and, in general, any}$$

number (N), consisting of (m) decimal places, is the same
as $\frac{N}{10^m}$

If a number ($\cdot A$) consisting of (m) decimal places be added to a number ($\cdot B$) consisting of ($m+n$) decimal places, their sum consists of ($m+n$) decimal places. For

$$\cdot A = \frac{A}{10^m}; \cdot B = \frac{B}{10^{m+n}}; \text{ then } \cdot A + \cdot B = \frac{A}{10^m} + \frac{B}{10^{m+n}}$$

$$= \frac{A \times 10^n + B}{10^{m+n}}; \text{ an expression which shews that } (n) \text{ cy-}$$

phers are added to $\cdot A$, and that ($m+n$) digits are pointed off from the sum of the quantities $\cdot A$ and $\cdot B$. Thus, to add together $3\cdot46572$ and $57\cdot38$, we add cyphers to the latter number to make its number of decimal places the same as in the former; and, after adding the numbers together, we point off as many decimals in the sum as each of the numbers contains.

If $\cdot A$ consist of (m) decimal places; and $\cdot B$ of (n) decimal places; their product consists of ($m+n$) decimal

places. For $\cdot A = \frac{A}{10^m}$, and $\cdot B = \frac{B}{10^n}$

$$\therefore \cdot A \times \cdot B = \frac{A}{10^m} \times \frac{B}{10^n} = \frac{AB}{10^{m+n}}, \text{ which is a}$$

number (AB) containing ($m+n$) decimal places.

In like manner $A \div B = \frac{A}{10^m} \div \frac{B}{10^n} = \frac{A}{10^m} \times \frac{10^n}{B} = \frac{A}{B \times 10^{m-n}}$, an expression which represents the quotient $\frac{A}{B}$ consisting of $(m-n)$ decimal places.

In a system of notation, where the local value is (x) , any number divided by $(x-1)$ leaves the same remainder as results from dividing the sum of its digits by $(x-1)$.

Let the digits of the given number be $a, b, c, d \dots e, g$ then the number (N) is $gx^m + ex^{m-1} \dots + dx^2 + cx^2 + bx + a$
The sum of the digits $(S) = g + e \dots + d + c + b + a$

$$\therefore N - S = g(x^m - 1) + e(x^{m-1} - 1) \dots + d(x^2 - 1) + c(x^2 - 1) + b(x - 1)$$

every term of which expression is divisible by $(x-1)$.

$$\text{Also } N = S' + g(x^m - 1) + e(x^{m-1} - 1) \dots + d(x^2 - 1) + c(x^2 - 1) + b(x - 1)$$

Divide this equation by $(x-1)$;

The remainder, if any, must be the same on both sides; on one side it arises from dividing the *number* (N) by $(x-1)$; and on the other side it arises from dividing the *sum of the digits* (S) by $x-1$.

In the common or denary scale, $x=10$; and any number, as 75482, divided by 9, leaves a remainder the same as results from dividing $(7+5+4+3+2)$ by 9.

Upon this property is founded the rule for proving questions in Multiplication, by casting out the nines.





Thus, if the multiplicand be $9a + b$
multiplier $9m + n$



product is $81am + 9(mb + na) + bn$

If the multiplicand, or the sum of its digits, be divided by 9, the remainder is (b); if the multiplier, or the sum of its digits, be divided by 9, the remainder is (n); and if the product, or the sum of its digits, be divided by 9, the remainder is (bn).

This property of 9 is also applicable to the number (3), because 3 is a measure of 9.

Having given a number in one system of notation, to find an equivalent number in another system; let $a, b, c, \dots d, e$, be the digits of one number, where the local value is (x), and let $A, B, C, \dots D, E$, be the digits of an equivalent number, where the local value is (y).

Then $ex^m + dx^{m-1} \dots + cx^2 + bx + a = Ey^n + Dy^{n-1} \dots + Cy^2 + By + A (=N)$

Divide both sides successively by (x)

Then $\frac{N}{x} = ex^{m-1} + dx^{m-2} \dots + cx + b \dots \dots \dots$ remainder (a).

Again divide by (x)

And $\frac{N}{x^2} = ex^{m-2} + dx^{m-3} \dots + c \dots \dots \dots$ remainder (b).

By continuing this process we have the successive remainders c, d, e : that is, if we continually divide a number $Ey^n + Dy^{n-1} \dots + Cy^2 + By + A$ by the local (x) of another system, the successive remainders are the digits of an equivalent number in the latter system.

Having given the number 75432 in a system of notation whose local value is 8, find the equivalent number in our own system.

The proposed number is $7 \times 8^4 + 5 \times 8^3 + 4 \times 8^2 + 3 \times 8 + 2$ or $28672 + 2560 + 256 + 24 + 2$; =

Or we may divide the proposed number successively by 10, employing the local value 8 in the same manner as we employ 10 in the common system.

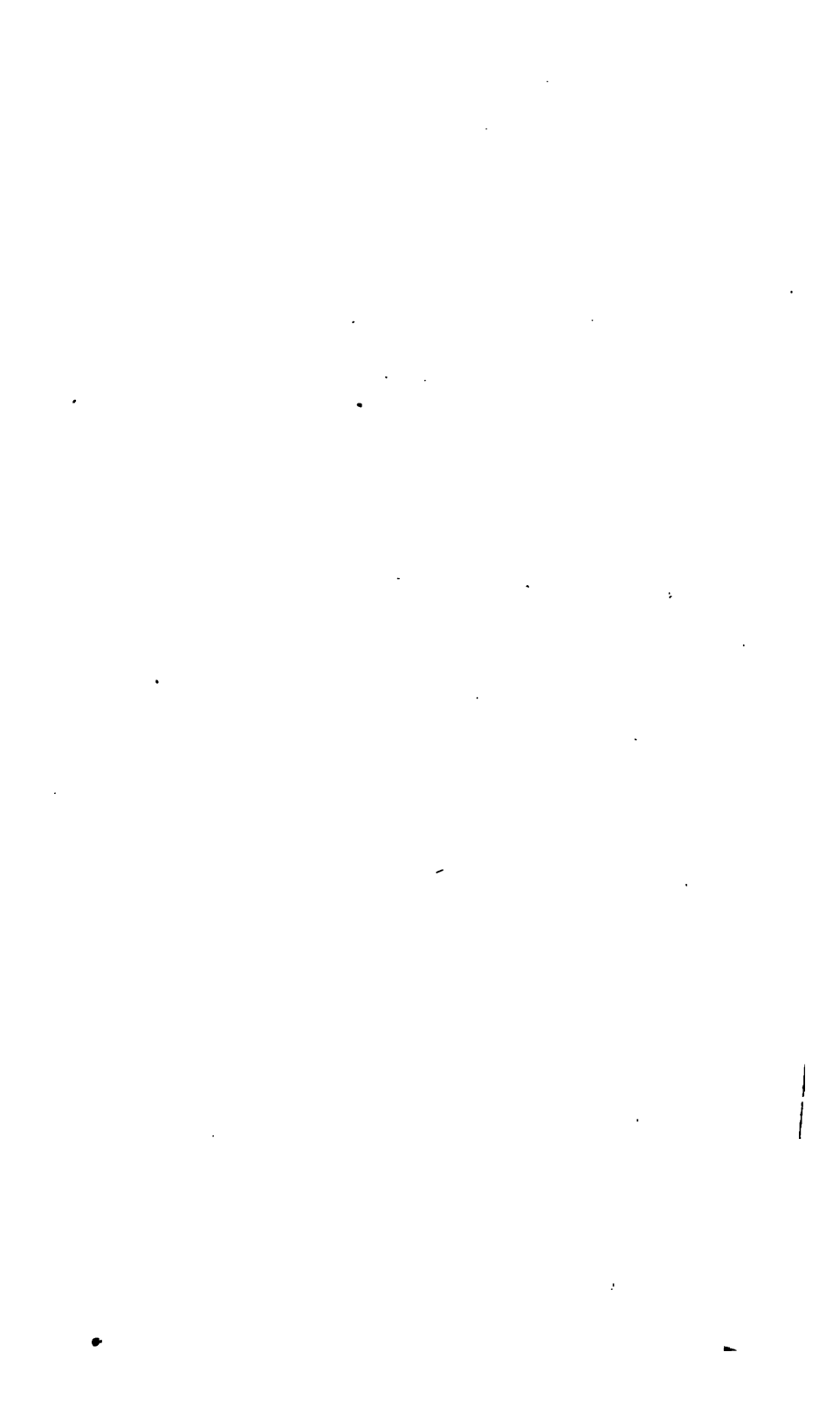
$$\begin{array}{r}
 10 \overline{) 75432} \\
 \underline{10 \overline{) 6117} \dots 4} \\
 \underline{10 \overline{) 473} \dots 1} \\
 \underline{10 \overline{) 37} \dots 5} \\
 \underline{10 \overline{) 3} \dots 1} \\
 0 \dots 3
 \end{array}$$

The number sought is therefore 31514, as before.

Having given the number 63542 in a system of notation whose local value is 7, find the equivalent number in a system where the local value is 8.

$$\begin{array}{r}
 8 \overline{) 63542} \\
 \underline{8 \overline{) 5503} \dots 6} \\
 \underline{8 \overline{) 500} \dots 3} \\
 \underline{8 \overline{) 42} \dots 5} \\
 \underline{8 \overline{) 3} \dots 6} \\
 0 \dots 3
 \end{array}$$

The number required therefore is 36536.



* In reckoning the divisors of a perfect number it is evident that the number itself must not be reckoned as one of them ($2^n - 1$) or else the definition of perfect number would be, "one which is twice the sum of all its divisors."

A PRIME number is one which admits of no divisor but unity : thus 3, 5, 7, 11, 13, are prime numbers. Two numbers are prime to each other when they admit no common divisor but unity : thus 8 and 9 are prime to each other.

A perfect number is one which is equal to the sum of all its divisors : thus 6, 28, 496, are perfect numbers.

If we take such a number of terms of the geometrical series $1 + 2 + 4 + 8 + 16$ as will make their sum $(2^n - 1)$ a prime number ; this sum multiplied by the n^{th} term of the series will be a *perfect number* : that is, $2^{n-1} \times (2^n - 1)$ is a *perfect number*, when (n) is of such a value as to make $(2^n - 1)$ a prime number. If we take $n=2$, $2^n - 1 = 3$, a prime number, and $2^{n-1} \times (2^n - 1) = 2 \times 3 = 6$, a perfect number. If we take $n=3$, $2^n - 1 = 7$, a prime number ; and $2^{n-1} \times (2^n - 1) = 4 \times 7 = 28$, a perfect number.

Now it may be proved that the sum of all the divisors of $2^{n-1} \times (2^n - 1)$ is equal to the quantity itself.

Since $2^n - 1$ is, by hypothesis, a prime number, suppose it = S .

Then all the divisors of $2^{n-1}S$ are 1, 2, 2^2 , 2^3 2^{n-1} and S ; $2S$, 2^2S , 2^3S $2^{n-1}S$. *

The sum of the first series of divisors is $2^n - 1$, or S ;

The sum of the second series is $S \times 2^{n-1} - S$; therefore the sum of all the divisors is $S + 2^{n-1}S - S = 2^{n-1}S = 2^{n-1}(2^n - 1)$; i. e. the sum of all the divisors of a quantity $2^{n-1} \times (2^n - 1)$ is equal to the quantity itself ; it is therefore a perfect number.

If we take $n = 5$, $2^n - 1 = 31$, a prime number ; $2^{n-1} = 16$, and $31 \times 16 = 496$, a perfect number.

If we take $n = 7$, $2^n - 1 = 127$, a prime number ; $2^{n-1} = 64$, and $127 \times 64 = 8128$, a perfect number.

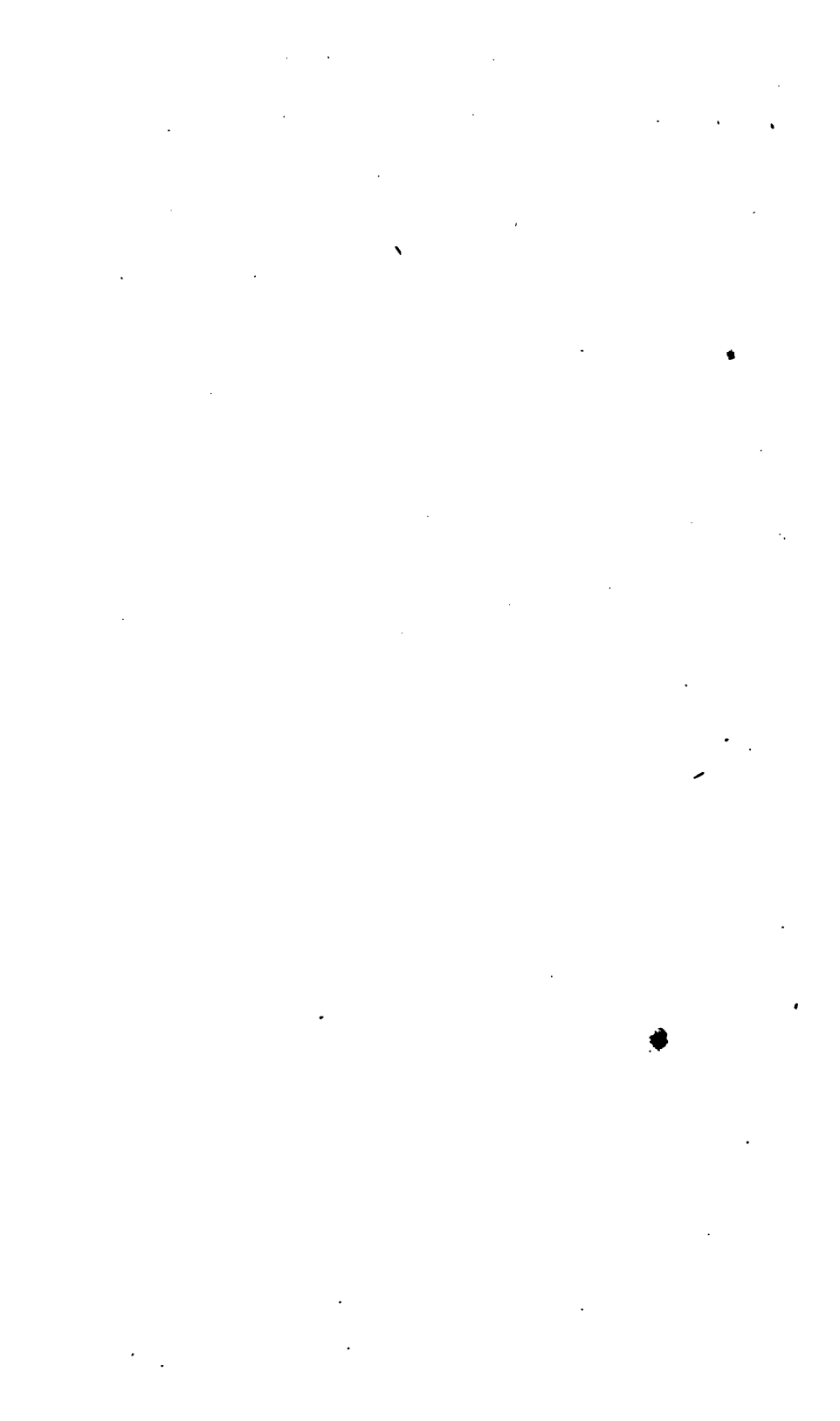
If we take $n = 13$, $2^n - 1 = 8191$, a prime number ; $2^{n-1} = 4096$, and $8191 \times 4096 = 33550336$.

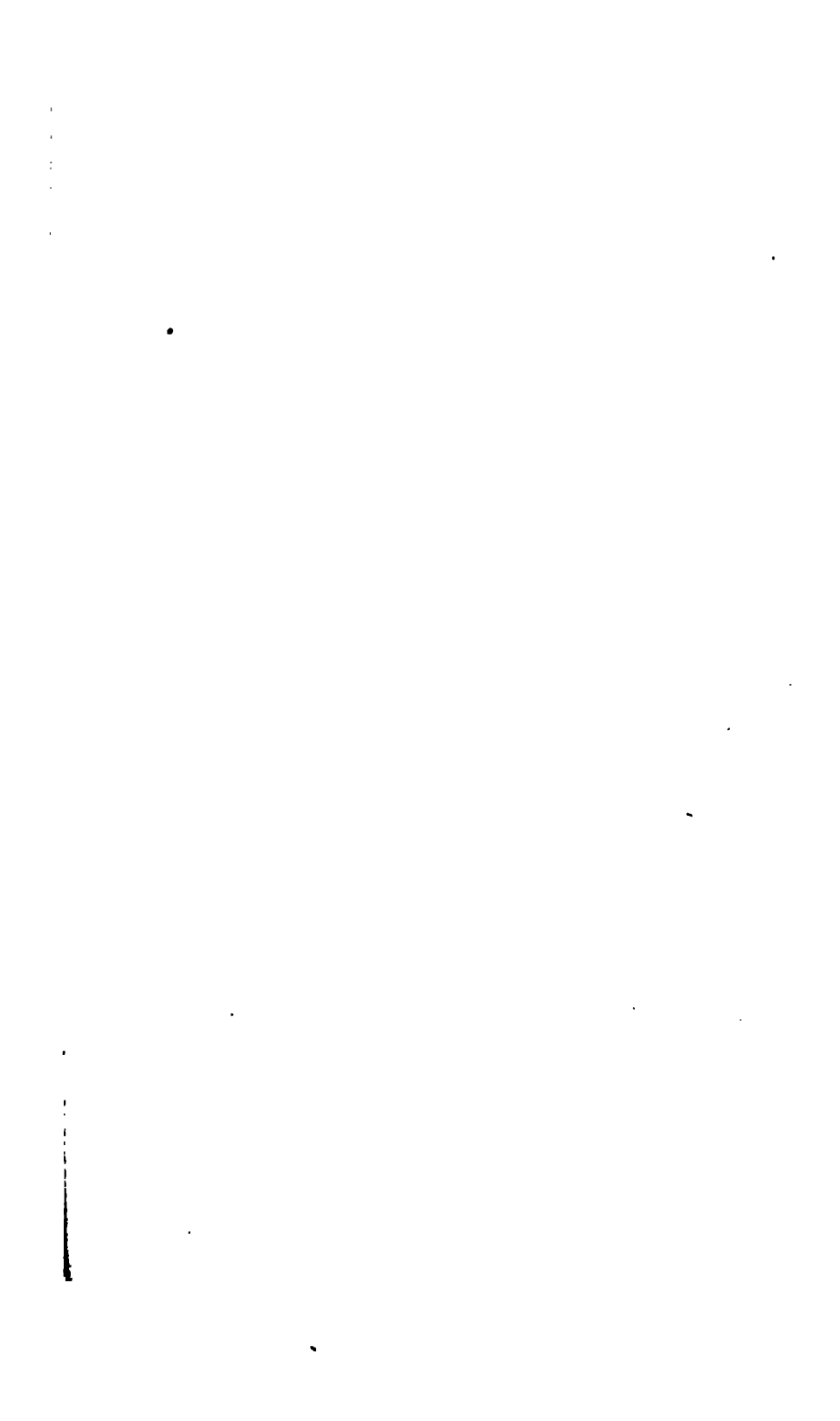
Every perfect number may be found by this theorem.

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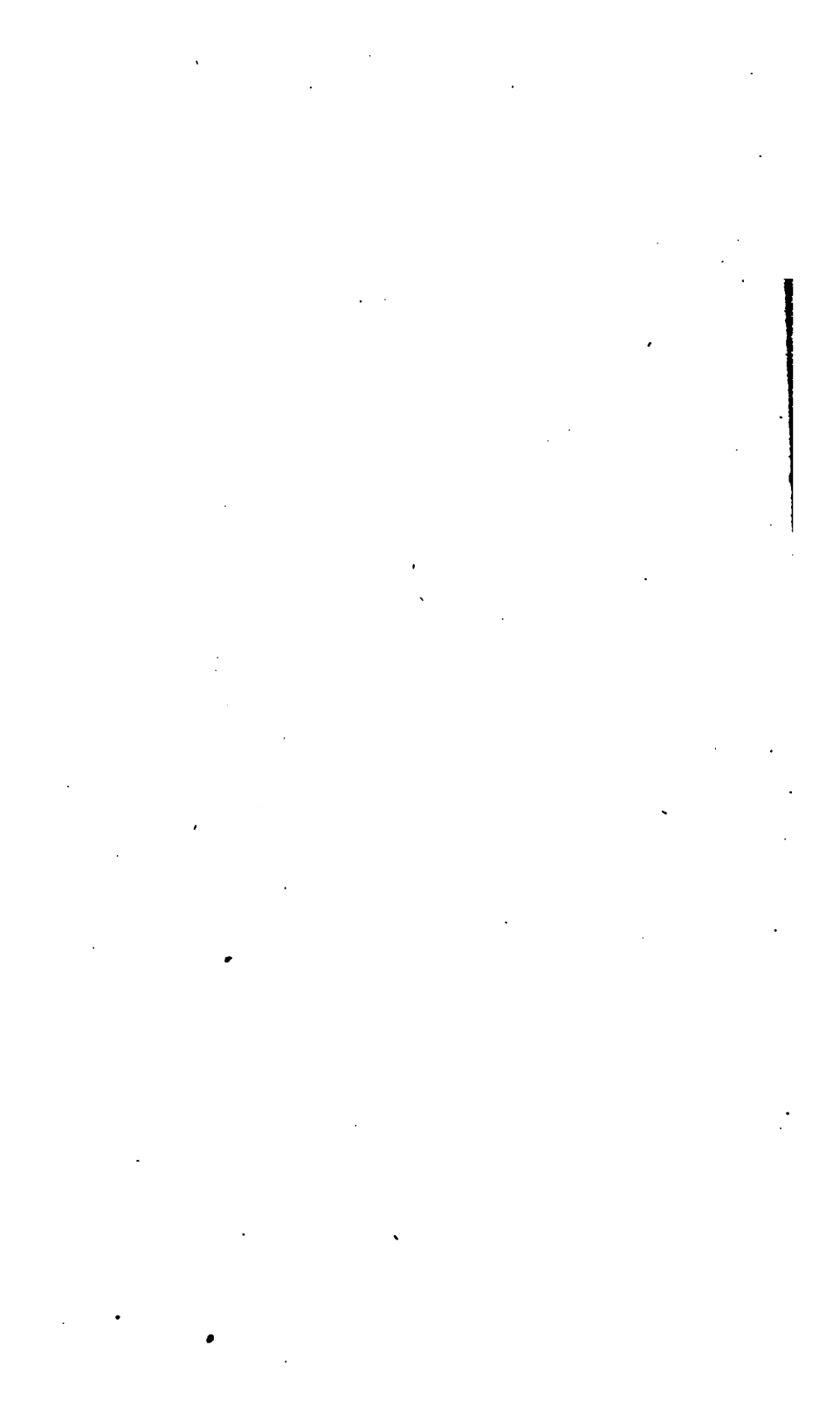
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PART II.





PART II.

SIMPLE EQUATIONS,

INVOLVING THE FIRST RULES OF ALGEBRA.

AN Equation is a proposition consisting of known and unknown quantities, connected together by the sign of equality; and it is called a *Simple* Equation when the index of the unknown quantity is unity.

The Solution of Equations depends chiefly on the four following axioms. 1. If equals be added to equals, the whole are equal. 2. If equals be taken from equals, the remainders are equal. 3. If equals be multiplied by equals, the products are equal. 4. If equals be divided by equals, the quotients are equal.

EXAMPLES.

1. Given $x + 4 = 7$; to find the value of x .

In order to have one side of the equation involving (x) only, we must subtract 4 from both sides, and the equation becomes $x + 4 - 4 = 7 - 4$

$$\text{or } x = 7 - 4 = 3.$$

This operation is evidently the same as if we had transposed 4 from one side of the equation to the other, and changed its sign.

2. Given $x-6=13$, to find x .

Here add 6 to both sides of the equation; or, with the same thing, transpose -6 and change its sign; we have

$$x=13+6=19.$$

3. Given $3x+2=2x-7+12$.

Transpose the quantities so as to have the terms containing (x) on one side of the equation, and the remaining terms on the other; then

$$3x-2x=-7+12-2$$

$$\text{or } x=12-9=3.$$

4. Given $5x-15=2x+6$; to find x .

$$5x-2x=6+15$$

$$\text{or } 3x=21$$

$$\therefore \frac{3x}{3} = \frac{21}{3} \quad (\text{Ax. 4})$$

$$\therefore x=7$$

5. Given $3x-6=x+18$; to find x .

$$3x-x=18+6$$

$$\text{Or } 2x=24$$

$$\therefore \frac{2x}{2} = \frac{24}{2} \quad (\text{Ax. 4})$$

$$\text{or } x=12.$$

6. A's age is double of B's; B's age is triple of C's; and the sum of all their ages is 140; find the age of each.

Let $x = \text{C's age}$

$\therefore 3x = \text{B's age}$

$\therefore 6x = \text{A's age}$

$$\therefore x+3x+6x=140;$$

$$\therefore 10x=140$$

$$\therefore \frac{10x}{10} = \frac{140}{10}$$

or $x=14$, C's age.

$3x=42$, B's age.

$6x=84$, A's age.





7. £300 was divided among A, B, and C; so that A received twice as much as B; and C as much as A and B together. How much did each receive?

Suppose B received x £

∴ A received $2x$ £

C received $3x$ £

$$\therefore x + 2x + 3x = 300$$

$$\text{or } 6x = 300$$

$$\therefore \frac{6x}{6} = \frac{300}{6}$$

or $x = \text{£}50$. B's share.

$2x = \text{£}100$. A's share.

$3x = \text{£}150$. C's share.

8. My purse and money are together worth 30 shillings, and the money is worth 5 times as much as the purse; how much money is in the purse?

Let x = value of the purse

∴ $5x$ = amount of the money

$$\therefore x + 5x = 30$$

$$\text{or } 6x = 30$$

$$\therefore \frac{6x}{6} = \frac{30}{6}$$

$$\therefore x = 5.$$

9. Two travellers started at the same time from London and York, and travel till they meet; one of them walks 16 miles a day, the other 14 miles a day; when will they meet, supposing the distance to be 197 miles?

Let x = number of days before they meet;

Then one traveller walks $16x$ miles,

and the other $14x$

$$\therefore 16x + 14x = 197$$

$$\text{or } 30x = 197$$

$$\therefore \frac{30x}{30} = \frac{197}{30}$$

$$\text{or } x = 6\frac{1}{3}.$$

10. *A* and *B* lay out equal sums of money in trade; when *A* has gained £126 in business, and *B* has lost £87, *A*'s money is found to be double of *B*'s; what did each lay out?

$$\begin{aligned}\text{Let } x &= \text{sum laid out by each,} \\ x + 126 &= A\text{'s property at last} \\ x - 87 &= B\text{'s do.} \\ \therefore 2(x - 87) &= x + 126 \\ \text{or } 2x - 174 &= x + 126 \\ \therefore x &= 126 + 174 = 300.\end{aligned}$$

11. There is a fish whose tail weighs 9lb., its head weighs as much as its tail and half its body, and its body weighs as much as its head and tail; find the weight of the fish.

$$\begin{aligned}\text{Let } x &= \text{weight of the head,} \\ 9 &= \text{weight of the tail,} \\ x + 9 &= \text{weight of the body,} \\ \therefore x &= 9 + \frac{x + 9}{2}\end{aligned}$$

Multiply both sides of the equation by 2;

$$\begin{aligned}2x &= 18 + x + 9 \\ \therefore 2x - x &= 18 + 9 = 27 \\ \text{or } x &= 27, \text{ weight of the head;} \\ \therefore \text{weight of the whole fish} &= x + 9 + x + 9 = 2x + 18 = 72.\end{aligned}$$

12. A person wished to give 3*d.* a piece to some beggars, but found he had not money enough by 8*d.* he therefore gave 2*d.* to each of them, and then had 3*d.* left; how many beggars were there?

$$\begin{aligned}\text{Let } x &= \text{number required,} \\ 3x - 8 &\text{represents the pence the person had;} \\ \text{also } 2x + 3 &\text{represents the same;} \\ \therefore 3x - 8 &= 2x + 3 \\ \therefore 3x - 2x &= 8 + 3 \\ \therefore x &= 11.\end{aligned}$$





13. A person left £560 between his son and daughter in such a manner that for every shilling the daughter received, the son should receive half a crown; what were their respective shares?

For every two sixpences received by the daughter, five sixpences are received by the son; therefore if we make

$2x$ the whole received by the daughter,

$5x$ is the sum received by the son;

$$\therefore 2x + 5x = 560$$

$$\text{or } 7x = 560$$

$$\therefore \frac{7x}{7} = \frac{560}{7}$$

$$\therefore x = 80$$

$\therefore$ daughter's portion is $2x = \text{£}160$

son's portion is $5x = \text{£}400$.

14. A is desired to think of a number: to this number he adds 19; from their sum he subtracts 50; he then adds 7; afterwards he subtracts 60; lastly he adds 6; and then finds his number to amount to 22: what was the number he first thought of?

Let $x =$ the number sought;

$$\therefore x + 19 - 50 + 7 - 60 + 6 = 22$$

$$\text{or } x + 32 - 110 = 22$$

$$\therefore x = 110 + 22 - 32 = 100.$$

15. How many apples has a fruiterer, supposing that if after you add 5 to his present number, and then subtract 3, and afterwards add 16, and next subtract 50, and lastly add 54, you find his number to be 43?

Let $x =$ the number he had at first.

Answer, 21.

PROOF OF THE RULE FOR SUBTRACTION.

IF it be required to subtract (b) from (a), the remainder is evidently $a - b$. If again it be required to subtract ($b - c$) from (a); since in this case the subtrahend is less by (c) than before, the remainder must be greater by (c) than before; but in the former case the remainder was $a - b$, therefore, in the latter case, the remainder is $a - b + c$; i.e.

$$\begin{array}{r} \text{If from } a \\ \text{we subtract } b - c \\ \hline \text{Diff. is } a - b + c \end{array}$$

SIMPLE EQUATIONS, INVOLVING THE FIRST RULES OF ALGEBRA.

EXAMPLES.

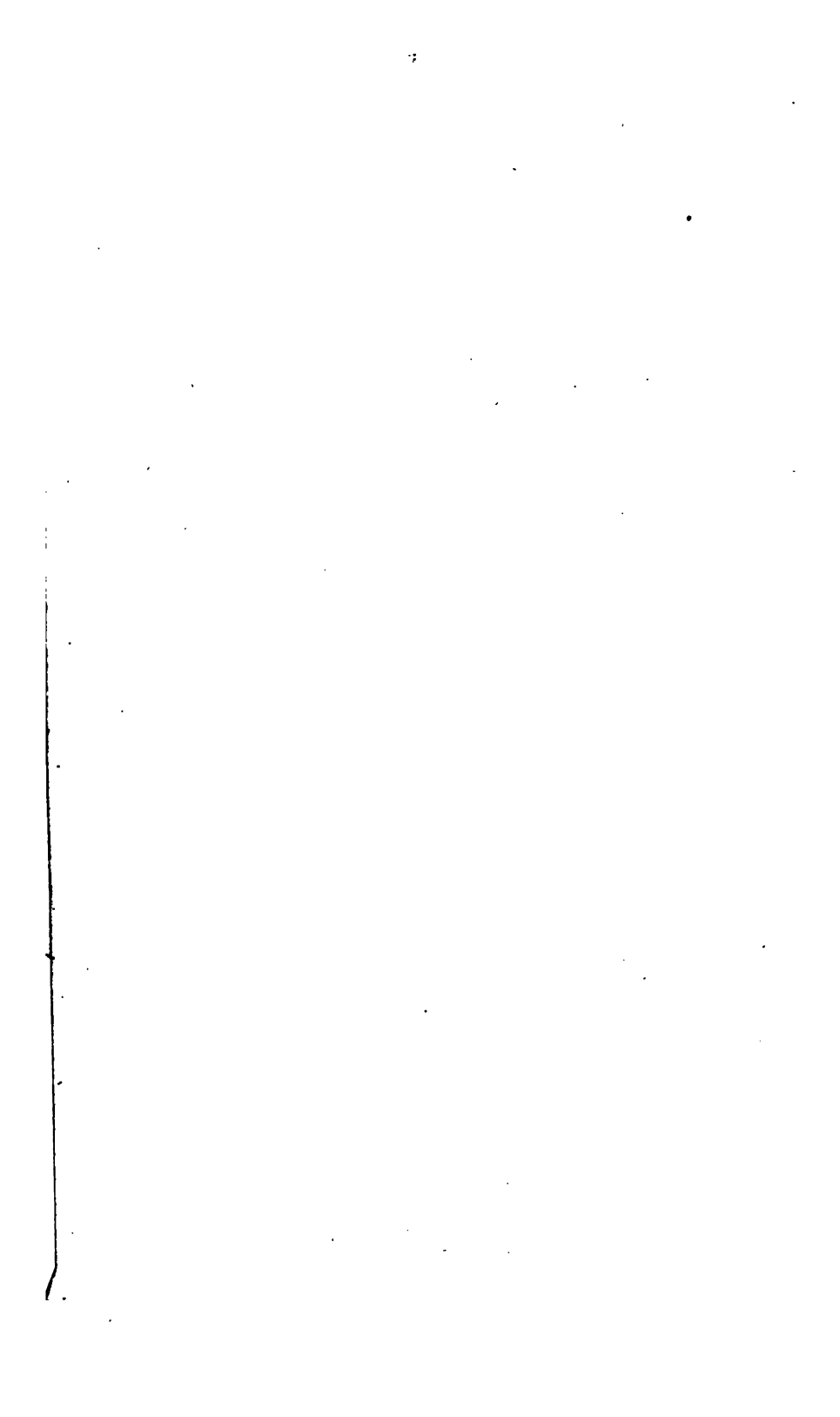
1. After paying away one fourth of my money, and then one fifth of the remainder, I had 72 guineas left; how much had I at first?

$$\begin{array}{rcl} \text{Let the number of guineas be } 4 \times 5 \times x & = & 20x \\ \text{Then one fourth of this sum} & = & 5x \end{array}$$

$$\begin{array}{rcl} \text{The remainder} & = & 15x \\ \text{And one fifth of this} & = & 3x \end{array}$$

$$\begin{array}{rcl} \text{Remaining sum} & = & 12x \\ \therefore 12x = 72 & \therefore x = 6, \text{ and } 20x = 120. \end{array}$$





2. At an election 1296 persons voted, and the successful candidate had a majority of 120; what number voted for each?

Let x = the number of voters on one side,
 $\therefore 1296 - x =$ do. on the other;

From $1296 - x$
 Take $+x$

$$\hline 1296 - 2x = 120.$$

$$\therefore 1176 = 2x \quad \therefore x = 588.$$

3. Divide the number 19 into two such parts, that ten times the one may exceed nine times the other by 19.

Let x = one part,
 $\therefore 19 - x =$ the other part;

$10 \times (19 - x) = 190 - 10x =$ ten times one part,
 $9x =$ nine times the other,

$$\hline \text{Difference } 190 - 19x = 19$$

$$\therefore 171 = 19x \quad \therefore x = 9$$

$$19 - x = 10.$$

4. In a mixture of copper, tin, and lead, $\frac{1}{3}$ of the whole, *minus* 16lb., was copper; $\frac{1}{3}$ of the whole, *minus* 12lb., was tin; and $\frac{1}{3}$ of the whole, *plus* 4lb., was lead; what quantity of each metal was in the mixture?

Suppose the mixture altogether contained $2 \times 3 \times 4 \times x$ ($= 24x$) lbs.

Then $12x - 16$ represents the copper,

$8x - 12$ tin,

$6x + 4$ lead;

$$\therefore 26x - 24 = 24x$$

$$\therefore 2x = 24; x = 12$$

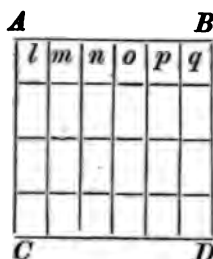
$\therefore$ there were 128lbs. of copper,

..... 84 tin,

..... 76 lead.

MULTIPLICATION.

WHETHER the quantity (a) be multiplied by (b), or the quantity (b) be multiplied by (a), the product is precisely the same. For suppose a set of equal parallelograms l, m, n, o, p, q , placed in an horizontal line, as in the annexed figure, and let the number of them in the horizontal row be (a); and if (b) be the number of these horizontal rows, then (b) times (a) makes up the whole figure $ABDC$. But it is clear that (b) is the number of parallelograms in each of the vertical rows, and as there are (a) vertical rows, (a) times (b) also makes up the same figure $ABDC$. Hence it appears that the same figure $ABDC$ represents the result, whether (a) parallelograms be taken (b) times, or (b) parallelograms be taken (a) times.



The product of two quantities with unlike signs is negative. For if (a) be multiplied by (b) the product is (ab); but if it be required to multiply (a) by ($b-c$), since in this case the multiplier is less by (c) than in the former case, this product must be less by (c) times the multiplicand than the former one. But in the former case the product was ab , $\therefore$ in the latter case it must be $ab-ac$: that is (a) multiplied by $b-c = ab-ac$.



+ as this is true for any value of a & c
let $a = c = 0$. $\therefore \overline{a-b} \times \overline{c-d} = ac - bc - da + db$
 $= +db$ or $-b \times -d = bd$.

By pursuing the same mode of reasoning, it may be shewn that the product of two negative quantities is positive. For having proved that $\overline{a-b} \times c = ac - bc$, let it be required to multiply $\overline{a-b}$ by $\overline{c-d}$; we have now diminished the multiplier by (d) , therefore the product ought to be less by (d) times the multiplicand than it was in the former case; that is (d) times $\overline{a-b}$ must be subtracted from $ac - bc$.

$$\begin{array}{r} \text{From} \quad ac - bc \\ \text{Take} \quad da - db \\ \hline \text{Remainder} \quad ac - bc - da + db \end{array}$$

✱ That is, $(a - b)$ multiplied by $(c - d)$ gives a product $ac - bc - da + bd$.

SIMPLE EQUATIONS.

Equations may involve as many unknown quantities as there are independent conditions; and it is often convenient to multiply one or both equations by such quantities as shall give the same co-efficient to the same unknown quantity in each. Thus,

$$\left. \begin{array}{l} \text{Given } x + 5y = 7 \\ \text{And } 2x + y = 5 \end{array} \right\} \text{ to find } x \text{ and } y.$$

Multiply the first equation by 2,

$$\begin{array}{r} \text{Then } 2x + 10y = 14 \\ 2x + y = 5 \\ \hline \end{array}$$

By Subtraction, $9y = 9 \therefore y = 1$

And $2x = 5 - y = 5 - 1 = 4 \therefore x = 2.$

$$\left. \begin{array}{l} \text{Given } 2x + 3y = 23 \\ \text{And } 5x - 2y = 10 \end{array} \right\} \text{ to find } x \text{ and } y$$

$$4x + 6y = 46$$

$$15x - 6y = 30$$

$$\text{By Addition, } 19x = 76 \quad \therefore x = \frac{76}{19} = 4$$

$$3y = 23 - 2x = 23 - 8 = 15$$

$$\therefore y = \frac{15}{3} = 5.$$

$$\left. \begin{array}{l} \text{Given } 5x + 4y = 58 \\ \text{And } 3x + 7y = 67 \end{array} \right\} \text{ to find } x \text{ and } y$$

$$15x + 12y = 174$$

$$15x + 35y = 335$$

$$23y = 161 \quad \therefore y = \frac{161}{23} = 7$$

$$3x = 67 - 7y = 67 - 49 = 18$$

$$\therefore x = \frac{18}{3} = 6.$$

$$\left. \begin{array}{l} \text{Given } ax + by = m \\ \text{And } cx + dy = n \end{array} \right\} \text{ to find } x \text{ and } y$$

$$cax + bcy = cm$$

$$cax + day = an$$

$$(bc - da) y = cm - an$$

$$y = \frac{cm - an}{bc - da}$$

$$ax = m - by; \quad \therefore x = \frac{m}{a} - \frac{b}{a} \times \frac{cm - an}{bc - da}$$



DIVISION.

SIMPLE EQUATIONS.

A person has a certain number (x) of apples; if $(x+3)$ be multiplied together three times, and this product be divided by (x^2+6x+9) the quotient is 10. What is the value of x ?

$$\overline{x+3} \times \overline{x+3} \times \overline{x+3} = x^3 + 9x^2 + 27x + 27.$$

$$\text{And } x^3 + 9x^2 + 27x + 27 \div (x^2 + 6x + 9) = x + 3$$

$$\therefore x + 3 = 10$$

$$\therefore x = 7.$$

A person thinks of a number, (x), he subtracts 4 from it; he multiplies the remainder three times by itself; he then divides this product by $(x-4)$; he next divides the quotient by $(x-4)$, and this second quotient is 9. What is the value of x ?

$$\overline{x-4} \times \overline{x-4} \times \overline{x-4} = x^3 - 12x^2 + 48x - 64$$

$$x^3 - 12x^2 + 48x - 64 \div (x-4) = x^2 - 8x + 16$$

$$\text{And } x^2 - 8x + 16 \div (x-4) = x - 4$$

$$\therefore x - 4 = 9 \quad \therefore x = 13.$$

A person thinks of a number, (x), he multiplies this number by itself; from this last number he subtracts 16; he then divides the remainder by $x-4$, and the quotient is 12. What is the number he thought of?

$$x^2 - 16 \div (x-4) = x + 4$$

$$\therefore x + 4 = 12$$

$$\therefore x = 8.$$

A person thinks of a number, he multiplies it by 7, he then increases this product by 3; then he divides this number by 2; lastly, he subtracts 4, and has 15 remaining. Find the number he thought of.

Let x = the number he thought of.

$$\frac{7x+3}{2} - 4 = 15$$

multiply both sides by 2.

$$7x+3-8=30$$

$$7x=30+8-3=35$$

$$\therefore x = \frac{35}{7} = 5.$$

If I divide a certain number by 25, and subtract 3 from the quotient, and then add 51, and afterwards subtract 40, I shall have 12 remaining. Find the first number.

Let x = number sought.

$$\therefore \frac{x}{25} - 3 + 51 - 40 = 12$$

$$\therefore \frac{x}{25} = 55 - 51 = 4$$

Then multiplying both sides by 25

$$x = 100.$$

How much money have I in my pocket when the 4th and 5th parts together amount to 11s. : 3d.?

Let $20x$ denote the pence.

$$\therefore \text{the 4th part is } 5x$$

$$\text{the 5th part is } 4x$$

$$\therefore 9x = 135 \text{ pence}$$

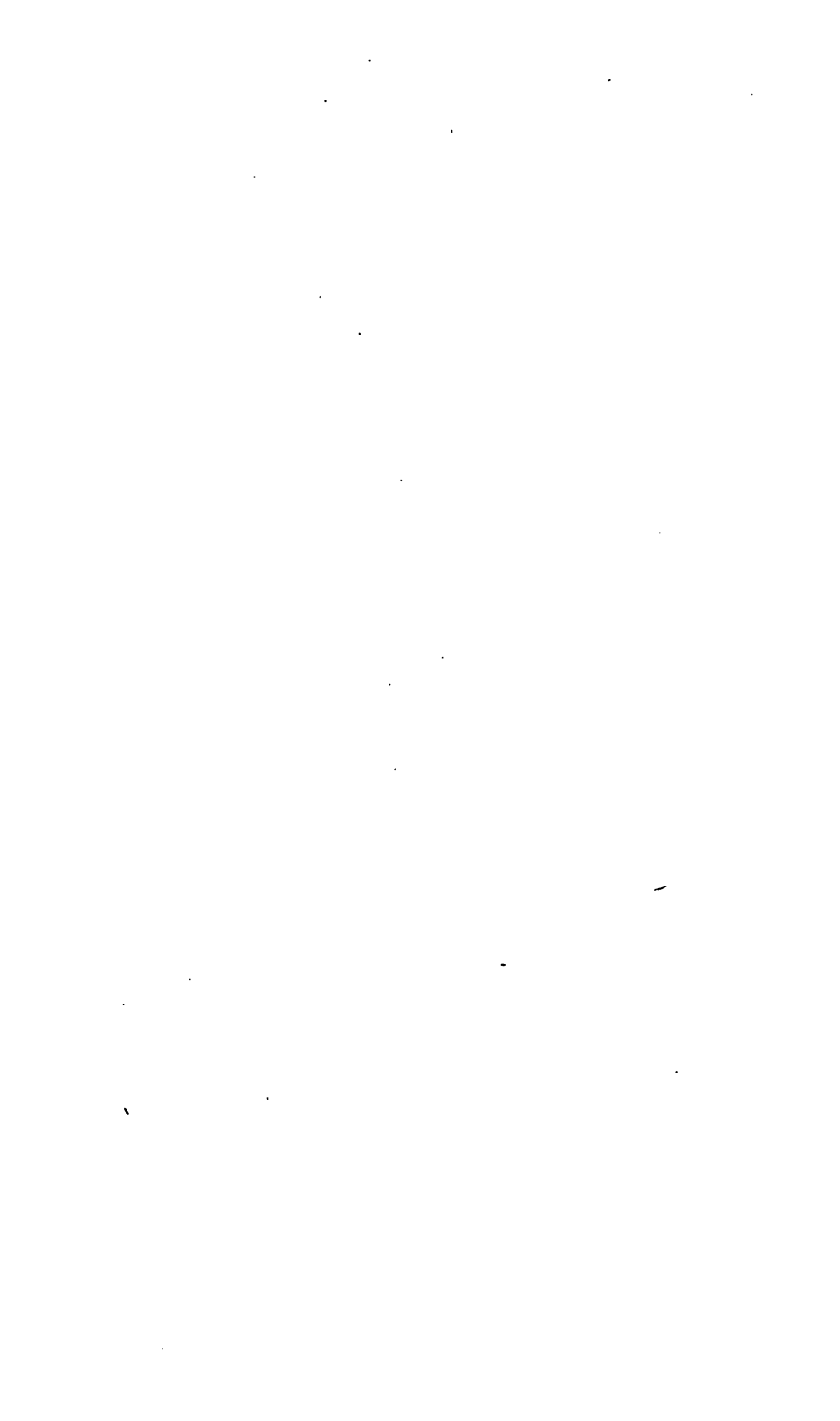
$$\therefore x = \frac{135}{9} = 15; \therefore 20x = 300d. = \text{£}1 : 5 : 0d.$$

Divide the number 46 into two such parts, that when the one is divided by 7, and the other by 3, the quotients may amount to 10. Find the parts.

Let x = one part.

$$\therefore 46 - x = \text{the other.}$$

$$\therefore \frac{46-x}{7} + \frac{x}{3} = 10$$





1st. multiply both sides by 7.

$$46 - x + \frac{7x}{3} = 70$$

$$138 - 3x + 7x = 210$$

$$\therefore 4x = 210 - 138 = 72$$

$$\therefore x = 18$$

$$46 - x = 28.$$

A person spends a third part of his income in board and lodging, an eighth part in clothes and washing, a tenth part in other expenses, and saves £218 yearly; what is his income?

Let $3 \times 8 \times 10x (=240x)$ = his income;

Then $\frac{240x}{3} = 80x$, one third of his income,

$\frac{240x}{8} = 30x$, an eighth of his income,

$\frac{240x}{10} = 24x$, a tenth of his income,

$$80x + 30x + 24x + 218 = 240x, \&c.$$

$$\therefore x = \frac{218}{106} = 2\frac{3}{53}$$

$$\therefore 240x = £493 : 11s. : 8\frac{1}{2}d.$$

A person has two boxes; if he put £8 into the first, it is then half as valuable as the second; but if he remove the £8 from the first to the second, it makes the latter worth three times as much as the first; find the worth of each box.

Let x = value of the first box,

$\therefore x + 8 \times 2$ = value of the second box,

$$\therefore x + 8 \times 2 + 8 = 3x$$

$$\therefore 2x + 16 + 8 = 3x \quad \therefore x = 24$$

$$\text{And } x + 8 \times 2 = 64.$$

A person thinks of a number (x), he multiplies this number by itself, from this last number he subtracts a given number (a^2); he then divides the remainder by $x-a$, and the quotient is $= b$; find x .

$$x^2 - a^2 \div (x-a) = b$$

$$\text{But } x^2 - a^2 \div (x-a) = x+a$$

$$\therefore x+a = b$$

$$\therefore x = b-a.$$

Divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by two, the third multiplied by 2, and the fourth divided by 2, the result in each case will be the same.

Let x = the third part,

$\therefore 2x$ = half the fourth,

$\therefore 4x$ = the fourth part;

Also $2x$ = first + 2 = second - 2,

$\therefore 2x-2$ = the first; and $2x+2$ = second,

$\therefore x+2x-2+2x+2+4x = 90 \quad \therefore \&c.$

$$x = 10.$$

Divide the number 36 into three such parts that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other.

Let $2x$ = first part,

$\therefore x = \frac{1}{2}$ of the second = $\frac{1}{4}$ of the third,

$\therefore 3x$ = the second, and $4x$ = the third;

$$\therefore 2x+3x+4x (=9x) = 36 \quad \therefore x = 4.$$

What is the amount of the dividend, when the quotient and remainder are each equal to 21, and the divisor is less than their sum by 7?

Let x represent the dividend,

$\therefore$ quotient = 21

remainder = 21

$42-7 (=35)$ is the divisor,

$$\therefore 35 \times 21 + 21 = 756 = x.$$



FRACTIONS.

the numerator and denominator of a fraction be either multiplied or divided by any quantity, the value of the fraction remains unchanged. Thus $\frac{a}{b}$ is the same as $\frac{ma}{mb}$.

the fraction $\frac{a}{b}$ denotes that unity is divided into (b) equal parts, and that (a) of these parts are taken. But if unity is divided into (mb) equal parts (as in the second fraction) each of those parts is $\frac{1}{m^{\text{th}}}$ as great as in the former case; therefore we must take (m) times as many of them as before, in order to have this latter fraction of same value as the former.

Upon this principle is founded the rule for reducing fractions to a common denominator, before they can be added or subtracted.

Thus to add together $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$;

$$\left. \begin{array}{l} a \times d \times f = adf \\ c \times b \times f = bcf \\ c \times b \times d = bcd \end{array} \right\} \text{ new numerators,}$$

$$b \times d \times f = bdf, \text{ new common denominator;}$$

and the new fractions are $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$, $\frac{bcd}{bdf}$, which are the

final fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, having their numerators and denominators respectively multiplied by the same quantity.

An integral quantity is multiplied by a fraction, either by multiplying the numerator, or dividing the denominator, of the fraction, by the integral quantity. Thus

$\frac{a}{b} \times c = \frac{ac}{b}$, or $\frac{a}{b \div c}$. For (c) times $\frac{a}{b}$ is the same as

$\frac{a}{b}$ added (c) times to itself, thus $\frac{a}{b} + \frac{a}{b} + \frac{a}{b} \dots$ to (c)

terms, which is $\frac{ca}{b}$. Again, of the two fractions $\frac{a}{b}$ and

$\frac{a}{b \div c}$, the latter is evidently (c) times as great as the

former; for in this latter case, unity is divided into $\frac{1}{c}$

the number of parts that it is in the former, therefore each of these parts must be (c) times as great as in the former case; and as the numerators are the same, and consequently the same number of parts taken in both cases, the latter fraction is (c) times as great as the former.

A fraction is divided by an integral quantity, either by dividing the numerator or multiplying the denominator by the integral quantity.

$\frac{a}{b} \div c$ is the same as $\frac{a \div c}{b}$; for in the two fractions $\frac{a}{b}$

and $\frac{a \div c}{b}$, unity is divided into the same number (b) of

parts; but in the latter case, $\frac{1}{c}$ th the number of those

parts are taken that are in the former case; therefore the

latter fraction is $\frac{1}{c}$ th ^{less than} ~~as great as~~ the former. Again $\frac{a}{b} \div c$

is the same as $\frac{a}{bc}$; for in this latter fraction unity is ~~di~~

vided into (c) times as many parts as in the fraction $\frac{a}{b}$;

and as the same number of parts are taken in both cases,

the value of $\frac{a}{bc}$ is $\frac{1}{c}$ th as great as $\frac{a}{b}$





Fractions are multiplied together by multiplying their numerators together, and likewise their denominators.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}; \text{ for (by what we have just proved)}$$

$$\frac{a}{b} \times c = \frac{ac}{b}; \text{ but this multiplier } (c) \text{ is } (d) \text{ times as great}$$

as the proposed one $\left(\frac{c}{d}\right)$; therefore the result $\frac{ac}{b}$ is (d)

times as great as the one required. The true product then must be $\frac{ac}{b}$ divided by (d) , or $\frac{ac}{bd}$.

The quotient of one fraction divided by another is found by inverting the divisor, and proceeding as in Multiplication.

$$\text{Thus } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}. \text{ For } \frac{a}{b} \div c =$$

$$\frac{a}{bc}; \text{ but this divisor } (c) \text{ is } (d) \text{ times as great as } \left(\frac{c}{d}\right)$$

therefore the quotient $\frac{a}{bc}$ is $\frac{1}{d}$ th part the quotient sought.

$$\text{The true quotient then will be } \frac{a}{bc} \times d = \frac{da}{bc}.$$

SIMPLE EQUATIONS,

INVOLVING FRACTIONS.

$$\text{Given } \frac{x}{7} + \frac{2x}{3} = 12 - \frac{3x}{4}.$$

Multiply both sides of the equation by $7 \times 3 \times 4 = 84$.

$$\frac{84x}{7} + \frac{84 \times 2x}{3} = 1008 - \frac{84 \times 3x}{4}$$

$$\text{or } 12x + 56x = 1008 - 63x$$

$$131x = 1008 \therefore x = \frac{1008}{131} = 7\frac{91}{131}.$$

Given $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$; to find x .

Multiply both sides by $2 \times 3 \times 4 = 24$

$$12x - 8x + 6x = 240, \text{ \&c.}$$

$$x = 24.$$

Given $x - 3 + \frac{2x}{3} = 40 - x + 19$; to find x .

$$2x + \frac{2x}{3} = 40 + 19 + 3 = 62.$$

$$\text{or } 6x + 2x = 186, \text{ \&c.}$$

$$x = 23\frac{1}{4}.$$

Given $a + x = \frac{x^2}{a+x}$; to find x .

$$a^2 + 2ax + x^2 = x^2 \quad \therefore 2ax = -a^2$$

$$\therefore 2x = -a \quad \therefore x = -\frac{a}{2}.$$

Given $\left. \begin{aligned} \frac{x}{2} + \frac{y}{3} &= 7 \\ \frac{x}{3} + \frac{y}{2} &= 8 \end{aligned} \right\} \text{ to find } x \text{ and } y$

$$3x + 2y = 42$$

$$2x + 3y = 48$$

$$\therefore 6x + 4y = 84$$

$$6x + 9y = 144$$

$$5y = 60, \therefore y = 12.$$



Given $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$; to find x .

Multiply both sides by $2 \times 3 \times 4 = 24$

$$12x - 8x + 6x = 240, \text{ \&c.} \quad x = 2$$

Given $x - 3 + \frac{2x}{3} = 40 - x + 19$; to find x .

$$2x + \frac{2x}{3} = 40 + 19 + 3 = 62.$$

$$\text{or } 6x + 2x = 186, \text{ \&c.} \quad x = 23$$

Given $a + x = \frac{x^2}{a+x}$; to find x .

$$a^2 + 2ax + x^2 = x^2 \quad \therefore 2ax = -a^2$$

$$\therefore 2x = -a \quad \therefore x = -\frac{a}{2}.$$

Given $\left. \begin{aligned} \frac{x}{2} + \frac{y}{3} &= 7 \\ \frac{x}{3} + \frac{y}{2} &= 8 \end{aligned} \right\} \text{ to find } x \text{ and } y$

$$3x + 2y = 42$$

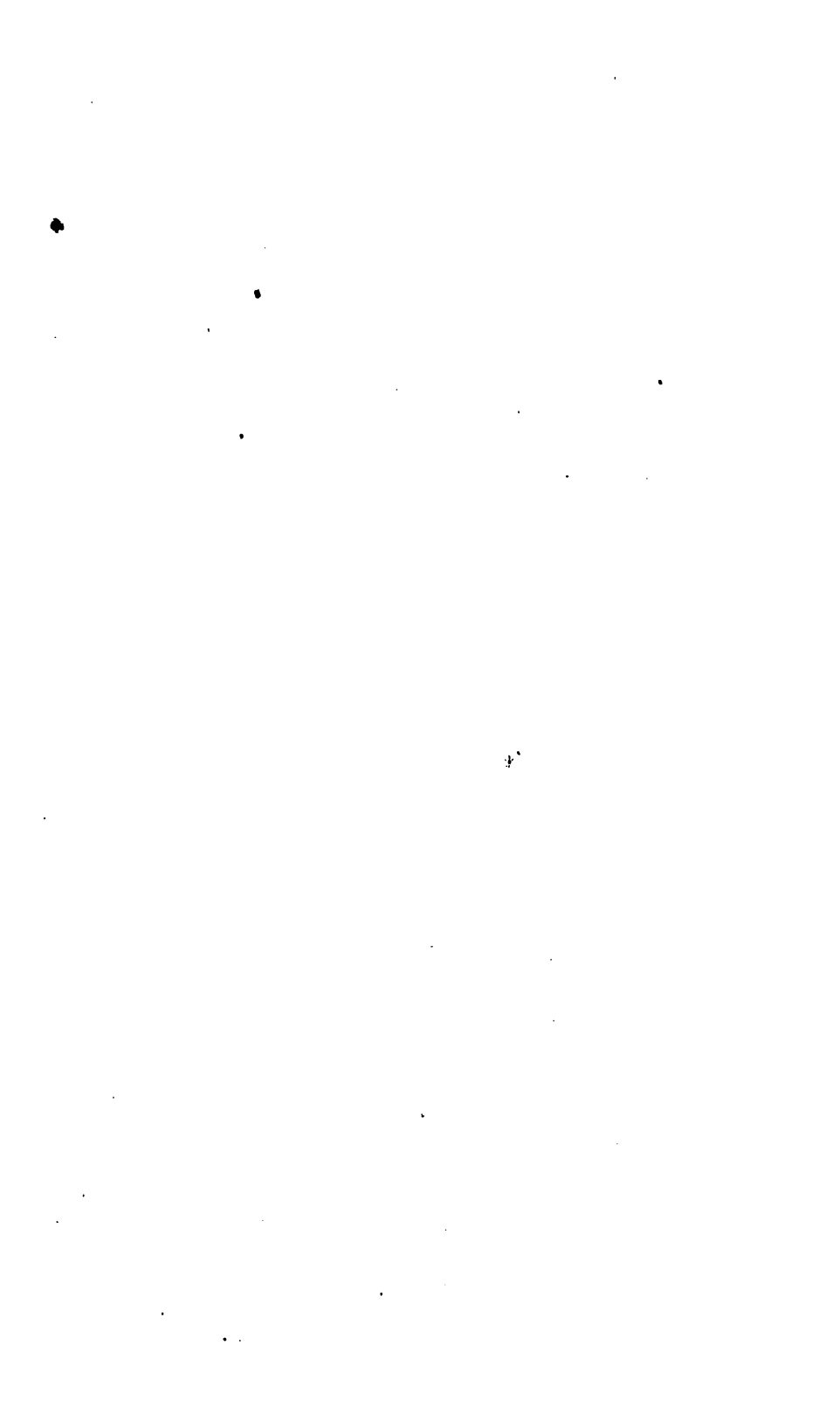
$$2x + 3y = 48$$

$$\therefore 6x + 4y = 84$$

$$6x + 9y = 144$$

$$5y = 60, \therefore y = 12.$$





Divide 36 into 3 such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be equal to each other.

Let the parts be x , y , and z .

$$\text{Then } \frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

$$\therefore x = \frac{2y}{3}$$

$$x + y + z = 36$$

$$z = \frac{4y}{3}$$

$$\text{or } \frac{2y}{3} + y + \frac{4y}{3} = 36.$$

$$2y + 3y + 4y = 108$$

$$9y = 108 \therefore y = 12$$

$$x = \frac{2y}{3} = 8; \quad z = \frac{4y}{3} = 16.$$

A, B, and C jointly buy goods for 30 shillings; their respective prices are as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Find the sums paid by each.

$$\text{Suppose A pays } \frac{x}{2}$$

$$\text{B } \frac{x}{3}$$

$$\text{C } \frac{x}{4}$$

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 30$$

$$12x + 8x + 6x = 720$$

$$26x = 720 \therefore x = \frac{720}{26} = \frac{360}{13} = 27s. : 8\frac{1}{2}d.$$

s. d.

$$\therefore \frac{x}{2} = 13 : 10\frac{1}{4}$$

$$\frac{x}{3} = 9 : 2\frac{1}{2} \text{ nearly.}$$

$$\frac{x}{4} = 6 : 11\frac{1}{4}.$$

A post is $\frac{1}{4}$ of its length in mud, $\frac{1}{3}$ in water, and 10 ft above the water; find its length.

Let x = the length.

$$\frac{x}{4} + \frac{x}{3} + 10 = x$$

$$3x + 4x + 120 = 12x$$

$$120 = 5x \quad \therefore x = \frac{120}{5} = 24 \text{ feet.}$$

Find such a number that if 5 be subtracted from it, $\frac{2}{3}$ the remainder will be 40.

Let x = the number.

$$\overline{x-5} \times \frac{2}{3} = 40;$$

$$\overline{x-5} \times 2 = 120$$

&c.

$$x = 65.$$

A person lost at cards $\frac{1}{4}$ of his money, and then back 3s.; the next game he lost $\frac{2}{3}$ of what he now had, then won back 12s.; in the next game he lost $\frac{1}{2}$ of what he then had left, and found at last that he had 24s. remaining. What had he at first?

x = number of shillings required.

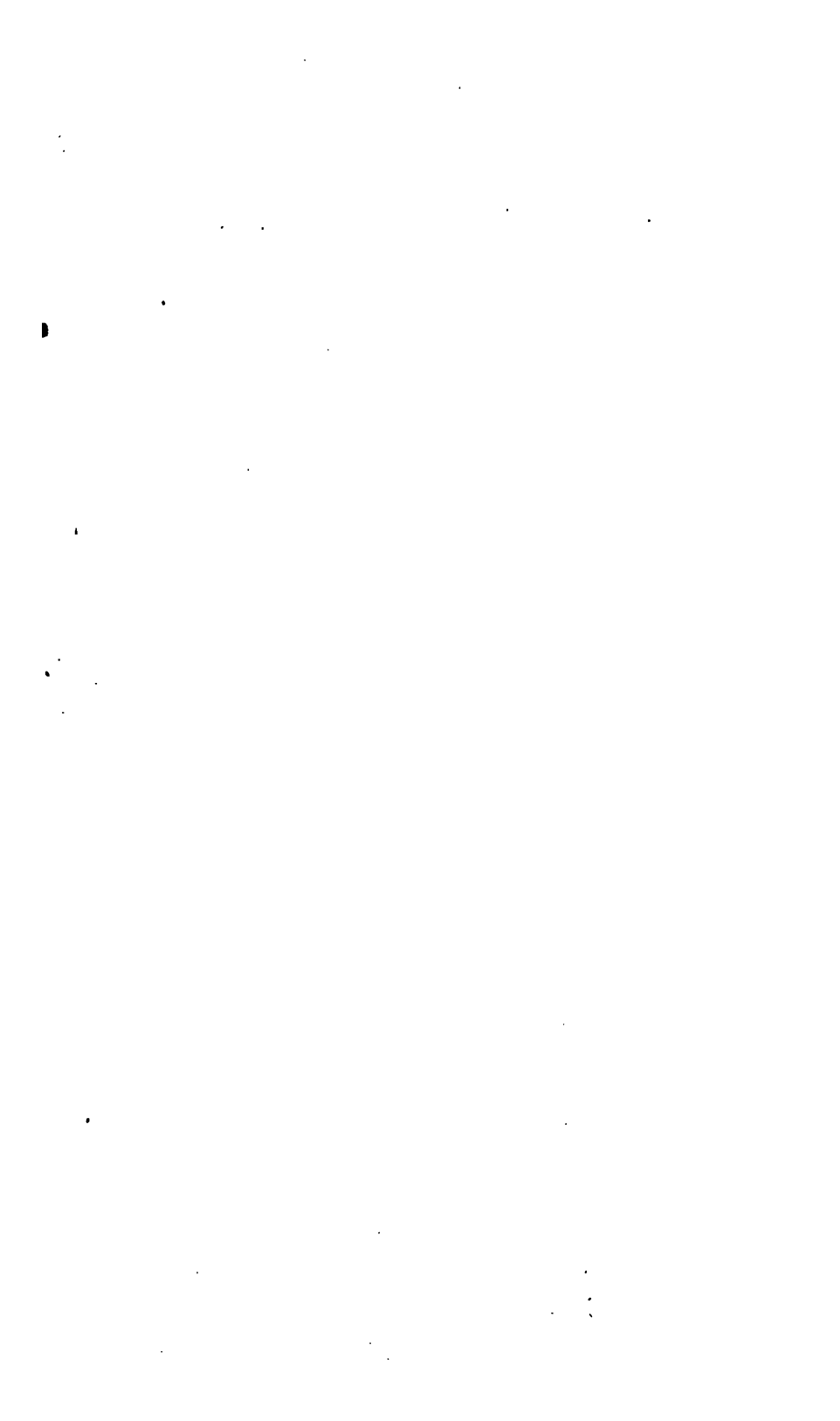
At the end of first game he had $\frac{3x}{4}$ remaining.

$\therefore \frac{3x}{4} + 3$ = his money at the end of second game.

$\frac{2}{3} \times \left(\frac{3x}{4} + 3 \right)$ = what he had remaining of it at the

third game, $= \frac{x}{2} + 2$;





$\therefore \frac{x}{2} + 2 + 12 (= \frac{x}{2} + 14) =$ his money at end of the next game.

$\frac{6}{7} (\frac{x}{2} + 14) = \frac{3x}{7} + 12 =$ what he then had remaining.

$\therefore \frac{3x}{7} + 12 = 24 \therefore \frac{3x}{7} = 12 \therefore x = 28.$

What number is that of which $\frac{2}{3}$ of $\frac{3}{4}$ added to $\frac{1}{2}$ of $\frac{5}{6}$ of it are equal to 11?

$x =$ number.

$$\frac{2x}{3} \times \frac{3}{4} = \frac{x}{2}; \frac{1}{2} \text{ of } \frac{5x}{6} = \frac{5x}{12}$$

$$\frac{x}{2} + \frac{5x}{12} = 11$$

$$6x + 5x = 132 \therefore 11x = 132$$

$$x = 12$$

A man and his wife usually drank a cask of beer in 12 days; but when the man was from home it lasted his wife 30 days. How long, at the usual rate, would it last the man alone?

$x =$ number of days required.

The man drinks $\frac{1}{x}$ th part of the beer in 1 day.

The wife $\frac{1}{30}$ th part of ditto in ditto.

They both drink $\frac{1}{12}$ th of it in 1 day.

$\therefore \frac{1}{x} + \frac{1}{30} = \frac{1}{12}$; multiply both sides of equation by $360x$; then $360 + 12x = 30x$

$$\therefore 360 = 18x \therefore x = 20.$$

A cistern has 3 pipes which can fill it in 3, 5, respectively. In what time will they fill it if they together?

Let x = number of hours required.

Then they all fill $\frac{1}{x}$ of the cistern in 1 hour.

$$\therefore \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{1}{x}$$

$$\therefore 35x + 21x + 15x = 105$$

$$\therefore x = \frac{105}{71} = 1\frac{34}{71}$$

Hours.	Min.	Sec.
= 1	: 28	: 43 $\frac{6}{71}$

A performs a piece of work in (a) days; B performs same in (b) days. In what time can they both perform?

x = time required.

A performs $\frac{1}{a}$ th of the work in 1 day.

B $\frac{1}{b}$ th of ditto in ditto.

A and B together perform $\frac{1}{x}$ of ditto in ditto.

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{x} \quad \therefore bx + ax = ab$$

$$\text{or } x(a+b) = ab \quad \therefore x = \frac{ab}{a+b}$$

Apply the above expression in the following example:
A man drinks a cask of beer in 20 days; his wife drinks in 30 days. How long will it last them both?

x = time required.

$$x = \frac{20 \times 30}{20 + 30} = \frac{600}{50} = 12.$$

We may also solve it thus, (direct method)

man's name:	days.	work
A	a	1
B	b	1
(A+B)	x	1

$$\therefore Aa = Bb \quad (1)$$

$$Aa = (A+B)x \quad (2)$$

$$Bb = (A+B)x \quad (3)$$

$\div$ Div (2) by A, or (3) by B

$$a = \left(1 + \frac{B}{A}\right)x$$

$$b = \left(1 + \frac{A}{B}\right)x$$

and putting for $\frac{A}{B}$ its value from (1),

$$a = \left(1 + \frac{a}{b}\right)x$$

$$b = \left(1 + \frac{b}{a}\right)x$$

$$\therefore x = \frac{ab}{a+b} \text{ from either.}$$



A and *B* together can perform a piece of work in eight days, *A* and *C* together in 9 days, and *B* and *C* together in 10 days: in how many days can each person perform the work?

Suppose *A* can do it alone in x days,

B y days,

C z days,

A does $\frac{1}{x}$ of the work in one day,

B $\frac{1}{y}$

C $\frac{1}{z}$

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{8}$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{9}$$

$$\frac{1}{y} - \frac{1}{z} = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$$

Also $\frac{1}{y} + \frac{1}{z} = \frac{1}{10}$

$$\therefore \frac{2}{y} = \frac{1}{10} + \frac{1}{72} = \frac{82}{720}$$

$$\therefore y = 17\frac{23}{41}, x = 14\frac{34}{41}, z = 23\frac{7}{41}.$$

Six gamblers *A*, *B*, *C*, *D*, *E*, *F*, sat down to play. *A* staked £68, *B* staked £73, *C* £83, *D* £86, *E* £84, and *F* £61. In consequence of some alarm, they all seized as much of the stakes as they could, and made their escape. On comparing the sums thus seized in the scramble, it appeared that if *A* gave up $\frac{1}{2}$ of the sum he had seized, *B* $\frac{1}{3}$, *C* $\frac{2}{5}$, *D* $\frac{1}{4}$, *E* $\frac{5}{6}$, and *F* $\frac{6}{7}$, of the sums they had respectively seized, and the amount thus surrendered was equally divided among them, then every one would have his own money again; how much did each man seize?

Suppose *A* seized *u*£, *B* *v*£ *C*
 D *x*£, *E* *y*£ *F*

$$\text{Let } \frac{u}{2} + \frac{2v}{3} + \frac{3w}{4} + \frac{4x}{5} + \frac{5y}{6} + \frac{6z}{7} =$$

$$\therefore \frac{u}{2} + \frac{Q}{6} = 68 \quad \therefore u = 136 - \frac{Q}{3}$$

$$\frac{v}{3} + \frac{Q}{6} = 73 \quad \therefore v = 219 - \frac{Q}{2}$$

$$\frac{w}{4} + \frac{Q}{6} = 83 \quad \therefore w = 332 - \frac{2Q}{3}$$

$$\frac{x}{5} + \frac{Q}{6} = 86 \quad \therefore x = 430 - \frac{5Q}{6}$$

$$\frac{y}{6} + \frac{Q}{6} = 84 \quad \therefore y = 504 - Q$$

$$\frac{z}{7} + \frac{Q}{6} = 61 \quad \therefore z = 427 - \frac{7Q}{6}$$

$$\therefore u + v + w + x + y + z = 2048 - \frac{27Q}{6}$$

$$\text{or } 455 = 2048 - \frac{27Q}{6}$$

$$\therefore \frac{27Q}{6} = 1593$$

$$\therefore \frac{9Q}{6} = 531 \quad \therefore Q = 354$$

$$\text{Hence } u = 136 - \frac{Q}{3} = 136 - \frac{354}{3} = \frac{54}{3} =$$

$$v = 219 - \frac{Q}{2} = 219 - 177 = 42$$

$$w = 332 - \frac{2Q}{3} = 332 - 236 = 96$$

$$x = 430 - \frac{5Q}{6} = 430 - 295 = 135$$

$$y = 504 - 354 = 150$$

$$z = 427 - \frac{7Q}{6} = 427 - 413 = 14.$$



EQUATIONS, INVOLVING

PROPORTION.

Given $x-y : y :: 1 : 2$ } to find x and y .
 And $xy=6$

First $x : y :: 3 : 2 \quad \therefore 2x=3y \quad \therefore x=\frac{3y}{2}$.

Substitute this value of x in the second equation;

then $\frac{3y^2}{2} = 6 \quad \therefore y^2 = 4 \quad \therefore y = 2$

$$x = \frac{3y}{2} = 3.$$

Given $x : x-y :: 2 : 1$ } to find x and y .
 And $x^2+y^2=20$

First $x : y :: 2 : 1 \quad \therefore x=2y$

$$\therefore x^2+y^2=4y^2+y^2=20$$

$$\text{or } 5y^2=20$$

$$\therefore y^2=4 \quad \therefore y=2$$

$$\text{And } x(=2y)=4$$

Given $\overline{x+y}^3 : \overline{x-y}^3 :: 27 : 1$ } to find x and y .
 $3xy=6$

First $x+y : x-y :: 3 : 1$

Then $x : y :: 2 : 1$

$\therefore$ &c.

Q 2

$$x=2, y=1.$$

Given $x^2 - y^2 : \overline{x-y}^3 :: 61 : 1$ } to find x and y
 $xy = 320$

$$x^2 - y^2 : \overline{x-y}^3 :: 61 : 1$$

$$\therefore 3x^2y - 3xy^2 : \overline{x-y}^3 :: 60 : 1$$

$$3xy(x-y) : \overline{x-y}^3 :: 60 : 1$$

$$\therefore 3xy : \overline{x-y}^2 :: 60 : 1$$

$$\therefore xy : \overline{x-y}^2 :: 20 : 1$$

$$\text{or } 320 : \overline{x-y}^2 :: 20 : 1$$

$$\therefore 16 : \overline{x-y}^2 :: 1 : 1$$

$$\therefore \overline{x-y}^2 = 16 \quad \therefore x-y=4$$

$$\therefore x^2 - 2xy + y^2 = 16$$

$$4xy = 1280$$

$$x^2 + 2xy + y^2 = 1296$$

$$\therefore x+y = 36$$

$$x-y = 4$$

$$2x=40 \quad \therefore x=20$$

$$2y=32 \quad \therefore y=16.$$

Two numbers are to each other as 2 to 3 ; but if added to each of them, their ratio becomes that of Find the numbers.

Let the numbers be $2x$ and $3x$.

Then $3x+5 : 2x+5 :: 7 : 5$

$$\therefore x : 2x+5 :: 2 : 5$$

$$\therefore 5x=4x+10$$

$$\therefore x=10$$

And the numbers are 20 and 30.



In one hour the minute hand gains on ^{hourly} ~~second~~ hand $55'$.

The two hands being together at 12 h. they will be together at $1^h 5' 27\frac{3}{11}"$ and at every multiple of $1^h 5' 27\frac{3}{11}"$

hand's rate	space described in minutes	x
62	$2 \times 60 + 2 \times 5 + x$	
1	$2 \times 5 + x$	

$$\therefore 12 : 1 :: 130 + x : 10 + x$$

$$\therefore 120 + 12x = 130 + x$$

$$\therefore x = \frac{10}{11}^m = 54'$$

$$\therefore \text{time} = 2^h \cdot 10^m \cdot 54\frac{6}{11}$$

The hour and minute hands of a clock are together at 12 o'clock; at what instant between 2 and 3 o'clock are they together?

Let x = the *part* of an hour required, between 2 and 3 o'clock.

Then since the minute hand moves 12 times as fast as the hour hand, $12 :: 1 :: 2 + x : x$

$$11 : 1 :: 2 : x$$

$$\therefore x = \frac{2}{11} = \frac{120'}{11} = 10' : 54'' \frac{6}{11}$$

Divide 100 into two such parts that the one increased by 10, may be to the other diminished by 10, as 3 to 2.

Let x = one part.

$$\therefore 100 - x = \text{the other.}$$

$$x + 10 : 100 - x - 10 :: 3 : 2$$

$$\therefore x + 10 : 100 :: 3 : 5$$

$$\therefore x + 10 : 20 :: 3 : 1$$

$$\therefore x + 10 = 60 \quad \therefore x = 50.$$

A footman agreed to serve his master for £8 a-year and *the price* of a livery; but was turned away at the end of 7 months, and received only £2 : 13s. : 4d. *the price of* his livery. What was the price of the livery?

Let x = price of the livery.

$$\text{Then } 12 : 7 :: x + 8 : x + 2\frac{2}{3}$$

$$:: x + 8 : x + \frac{8}{3}$$

$$:: 3x + 24 : 3x + 8$$

$$\therefore 5 : 7 :: 16 : 3x + 8$$

$$\therefore 15x + 40 = 112 \quad \therefore 15x = 72$$

$$x = 4\frac{4}{5} = £4 : 16s. : 0d.$$

A fruiterer bought apples of one kind at two a penny, and the same quantity of another kind at three a penny; and by selling the whole at five for 2*d.*, found himself a loser of 1*s.* What quantity did he buy?

Let x = quantity of each sort.

$$2 \cdot x :: 1 : \frac{x}{2}, \text{ price of } (x) \text{ apples at 2 a penny.}$$

$$3 : x :: 1 : \frac{x}{3}, \text{ price of } (x) \text{ apples at 3 a penny.}$$

$$5 : 2x :: 2 : \frac{4x}{5}, \text{ price of all at five for } 2d.$$

$$\therefore \frac{x}{2} + \frac{x}{3} = \frac{4x}{5} + 12$$

$$15x + 10x = 24x + 360$$

$$x = 360.$$

✱ A greyhound starts a hare when she is 50 of her own leaps a-head of him. She takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare?

Let $3x$ = leaps taken by the greyhound to catch the hare

$\therefore 4x$ = leaps taken by the hare in the same time.

Again, if we measure a *given space* in terms of their respective leaps, we have

$$3x : 4x + 50 :: 2 : 3$$

$$\therefore 9x = 8x + 100$$

$$\therefore x = 100, \text{ and } 3x = 300.$$

What two numbers are those whose difference, sum and product, are respectively as the numbers 2, 8, and 15?

Let x and y be the numbers.

$$\text{Then } x - y : x + y :: 2 : 8 :: 1 : 4$$

$$\therefore x : y :: 5 : 3 \quad \therefore x = \frac{5y}{3}$$

$$\therefore x + y : y :: 8 : 3 \quad \therefore x + y : 8 :: y : 3$$

$$x + y : xy :: 8 : 15$$

$$\frac{2y}{3} : \frac{5y}{3} :: 8 : 15$$

$$2 : 5 :: 8 : 15$$

no. of apples.	price at purchase.	Value
x	$\frac{1}{2}$	A
x	$\frac{1}{3}$	B
$2x$	$\frac{2}{5}$	$A +$

$$\begin{aligned}
 \text{hence } \frac{x}{2} &= A \\
 \frac{x}{3} &= B \\
 \frac{4x}{5} &= A + B - 12 \\
 &= \frac{5}{6}x - 12
 \end{aligned}$$

$$\therefore x = 360$$

We may be required in this problem find the lengths of the laps of the hare and dog respectively, the distance run over by the dog, and the number of laps back each. We have then

	Length of lap	Number laps.	Distance
u	H	h	A
v	D	d	$50H + A$

$$\text{wh. } Hh = A \quad (1)$$

$$Dd = 50H + A \quad (2)$$

$$\text{also } 2D = 3H \quad (3)$$

$$\frac{h}{d} = \frac{4}{3} \quad (4) \quad (\text{from})$$

$$2) - (2) \quad Dd - Hd = 50H$$

$$\text{or } \left(\frac{Dd}{Hh} - 1 \right) h = 50$$

$$3) \times (4) \quad \frac{8}{9} Dd = Hh \quad (5)$$

$$\therefore \left(\frac{8}{9} - 1 \right) h = 50$$

$$\therefore h = 400$$

$$\therefore Id = \frac{3}{4} h = 300$$

~~$$2 \text{ from } H = \frac{8 \cdot 8}{9} \frac{d}{h}$$~~

~~$$= \frac{8 \cdot 8 \cdot 3}{9 \cdot 4}$$~~

~~$$= \frac{64}{12}$$~~

$$\text{4 from (5)} \quad \frac{D}{4} = \frac{A}{h} = \frac{A}{400}$$

$$2) \text{ If } Dd = \frac{A}{20} + A = \frac{21}{20} A$$

$$\therefore D = \frac{21A}{20 \cdot 300} =$$

Also $x+y : xy :: 8 : 15 \therefore x+y : 8 :: xy : 15$

$$\therefore xy : 15 :: y : 3$$

$$\therefore x : 15 :: 1 : 3$$

$$\therefore x = \frac{15}{3} = 5$$

$$\text{and } y = 3.$$

Divide the number 18 into two such parts that the quotient of the greater part divided by the less may be to the quotient of the less divided by the greater, as 25 : 16.

Let $x =$ greater part.

$$\therefore 18-x = \text{the less}$$

$$\frac{x}{18-x} : \frac{18-x}{x} :: 25 : 16$$

$$\therefore x^2 : (18-x)^2 :: 25 : 16$$

$$\therefore x : 18-x :: 5 : 4$$

$$\therefore x : 18 :: 5 : 9$$

$$\therefore x : 2 :: 5 : 1 \quad \therefore x = 10$$

The numbers are 10 and 8.

A person has 2640 coins of two kinds; the number of one kind is $4\frac{1}{2}$ times as many as of the other; how many has he of each?

Let $x =$ the number of one kind;

$$\therefore 2640 - x = \text{the number of the other kind,}$$

$$\text{Then } x : 2640 - x :: 4\frac{1}{2} : 1$$

$$\therefore x : 2640 :: 4\frac{1}{2} : 5\frac{1}{2} :: 9 : 11$$

$$\therefore x : 240 :: 9 : 1 \quad \therefore x = 2160.$$

QUADRATIC EQUATIONS.

AN affected Quadratic Equation may be thus generally expressed, $x^2 \pm px \pm q = 0$.

1. Let $x^2 - px = -q$

$$\therefore x^2 - px + \frac{p^2}{4} = \frac{p^2}{4} - q$$

$$\therefore x - \frac{p}{2} = \pm \sqrt{\frac{p^2}{4} - q}$$

$$\therefore x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}; \text{ and the two}$$

values of x are $\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}$ and $\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$

2. Let $x^2 + px = -q$; then by proceeding as above

we find $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$

In both these cases it appears that the values of (x) are possible, only where $\frac{p^2}{4}$ is greater than q .

If the two values of (x) , in either case, be added together, their sum is $= p$, the co-efficient of (x) in the second term of the proposed equation.

Also if they be multiplied together, their product is equal to (q) , the third term in the original equation $x^2 \pm px + q = 0$;



In a quadratic equation $x^2 - px + q = 0$
 whose roots are a, b . $p = a+b$. & $q =$

for $a^2 - pa + q = 0$

& $b^2 - pb + q = 0$

$$\therefore \frac{a^2 - b^2 - a + b \cdot p}{a - b} = 0$$

$$\therefore p = \frac{a^2 - b^2}{a - b} = a + b$$

also by $+$ $a^2 + b^2 - a + b \cdot p + 2q = 0$

$$\text{or } a^2 + b^2 - a + b \cdot p + 2q = 0$$

$$\therefore -2ab + 2q = 0 \therefore q = ab.$$

$$(x-a)(x-b)(x-c) = x^3 - ax^2 - bx^2 - cx^2 + abx + acx + bcx - abc$$

$$= x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc.$$

If $n = 3$

$$p = a+b+c$$

$$q = ab+bc+ac$$

$$2 = abc$$

then $(x-a)(x-b)(x-c)$ to n factors (where x
~~is not equal to~~ $x=a, \forall x=b, \forall x=c$) $= x^n - px^{n-1} + qx^{n-2} -$

In which equation $p =$ sum of the roots

$q =$ sum of all combinations of
 taken 2 & 2 together.

$2 =$ the product of the roots.

$$\left(\pm \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right) \times \left(\pm \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right) = \frac{p^2}{4} - \left(\frac{p^2}{4} - q\right) = +q.$$

Any equation, as $x^2 - 5x + 10 = 4$, may be made to coincide with the equation $x^2 - px + q = 0$. Thus $x^2 - 5x + 6 = 0$; here $p = 5$, $q = 6$, and the two values of x are 3 and 2. It is also to be observed that $3 + 2 = 5$, the co-efficient of the second term, and $3 \times 2 = 6$, the third term in the proposed equation.

3. If the proposed equation be $x^2 - px = q$; by proceeding as before we find $x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$; if it be $x^2 + px = q$, $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$.

Given $x^{2n} \pm q = 0$, to find x .

$$x^{2n} = \mp q$$

$$\therefore x = \sqrt[n]{\mp q}$$

Given $x^{2n} + bx^n - q = 0$, to find x

Let $x^n = y$

$$y^2 + by = q;$$

$$y = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + q}$$

$$\therefore x = y^{\frac{1}{n}} = \sqrt[n]{-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + q}}.$$

QUESTIONS FOR PRACTICE.

Given $\left. \begin{array}{l} x^2 - xy = a \\ xy - y^2 = b \end{array} \right\}$ to find x and y

$$x^2 - 2xy + y^2 = a - b$$

$$\therefore x - y = \sqrt{a - b}$$

$$x \cdot \overline{x - y} = a \quad \therefore x = \frac{a}{\sqrt{a - b}}$$

$$y = \frac{a}{\sqrt{a - b}} - \sqrt{a - b}$$

$$= \frac{b}{\sqrt{a - b}}$$

Given $\left. \begin{array}{l} xy = a \\ x^2 + y^2 = b \end{array} \right\}$ to find x and y

$$x^2 + 2xy + y^2 = \sqrt{b + 2a}$$

$$x^2 - 2xy + y^2 = b - 2a$$

$$\therefore x - y = \sqrt{b - 2a}$$

$$x + y = \sqrt{b + 2a}$$

$$2x = \sqrt{b - 2a} + \sqrt{b + 2a}$$

$$2y = \sqrt{b + 2a} - \sqrt{b - 2a}$$

Given $\left. \begin{array}{l} x^2 - y^2 = a \\ x + y = b \end{array} \right\}$ to find x and y

$$x - y = \frac{a}{b}$$

$$x + y = b$$

$$\therefore 2x = b + \frac{a}{b}$$

$$2y = b - \frac{a}{b}, \text{ \&c.}$$





Given $\left. \begin{array}{l} x^2 + y^2 = a \\ x + y = b \end{array} \right\}$ to find x and y

$$\begin{array}{r} x^2 + 2xy + y^2 = b^2 \\ x^2 \quad \quad + \quad y^2 = a \end{array}$$

$$\begin{array}{r} 2xy = b^2 - a \\ x^2 \quad \quad + \quad y^2 = a \end{array}$$

$$x^2 - 2xy + y^2 = 2a - b^2$$

$$x - y = \sqrt{2a - b^2}$$

$$x + y = b, \text{ \&c.}$$

Given $\left. \begin{array}{l} x + y = a \\ xy = b \end{array} \right\}$ to find x and y .

$$(x + y)^2 = a^2$$

$$4xy = 4b$$

$$\therefore (x - y)^2 = a^2 - 4b, \text{ \&c.}$$

$$x = a + \sqrt{a^2 - 4b}$$

$$y = a - \sqrt{a^2 - 4b}.$$

Given $\left. \begin{array}{l} x^2 + xy = a \\ xy + y^2 = b \end{array} \right\}$ to find x and y

$$(x + y)^2 = a + b$$

$$x \cdot x + y = \text{ \&c.}$$

$$x = \frac{a}{\sqrt{a + b}}$$

$$y = \frac{b}{\sqrt{a + b}}.$$

The sum of two numbers is (a), and the sum of their squares is (b): find the numbers.

Let $x + y$ be the greater number.

$x - y$ be the less.

$$\therefore 2x = a \quad \therefore x = \frac{a}{2}$$

$$2(x^2 + y^2) = b; \text{ substitute } \frac{a}{2} \text{ for } x, \text{ \&c.}$$

The sum of two numbers is a , and the sum of their cubes is (b) .

Let $x+y$ be the greater number.

$x-y$ less.

$$2(x^3 + 3xy^2) = b; \quad x = \frac{a}{2}$$

Substitute for x , &c.

1. Divide 146 into two such parts that the difference of their square roots may be 6.

Let x = square root of one number,

$\therefore x+6$ = square root of the other, &c.

$x = 5$ and -11 .

2. The sum of two numbers added to the sum of their squares is $=338$; and the product of the numbers is $=156$: find them.

The numbers are 12 and 13.

3. Two travellers start at the same time from C and D and travel till they meet; when they meet it is found that A had travelled 30 miles more than B; and that A would at that rate, reach D in 4 days, and B reach C in 9 days: find the distance of C from D.

Suppose A to have travelled $x+15$ miles } when they
And B $x-15$ } meet.

$\therefore 2x$ is the distance between C and D.

Then A has to travel $(x-15)$ miles in 4 days.

And B has to travel $(x+15)$ miles in 9 days.

Hence we find the days they have already travelled, &c.

The distance sought is 150 miles.

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4. Divide 60 into two such parts that their product shall amount to 864.

The parts are 24 and 36.

5. The sum of two numbers is 28, and the sum of their squares is 400 : find them.

Answer, 12 and 16.

6. The difference of two numbers is 4, and the sum of their squares is 400 : find them.

Answer, 12 and 16.

7. The sum of two numbers is 17, and the sum of their cubes is 1343 : find them.

Let $x + y =$ greater number

$x - y =$ less number

Answer, 7 and 10.

8. A buys a number of oxen for 80 guineas ; but if he had bought 4 more for the same money, he would have bought them a guinea a-head cheaper. How many oxen did he buy ?

Answer, 16.

9. Find that number which, when added to its square root, will make 210.

Answer, 196.

10. What two numbers are those whose product is 192, and sum of whose squares is 640 ?

Answer, 8 and 24.

11. A pays £33 : 15s. : 0d. for cloth, which he sells again for 48s. per piece ; and gains as much as one piece cost him. What did he pay for one piece of cloth ?

Answer, 45s.

12. A had such a number of pounds that when subtracted from B's, the remainder was 10; and when multiplied by B's, the product was 75. Find the number belonging to each.

Answer, 15 and 5.

13. Find 2 numbers; the less² taken from the greater = 36; greater + 6 times the less = 148.

Let $x + y$ = greater

$x - y$ = less

one number is 100

the other 8.

14. If from 3 times the longest side of a parallelogram the square of the lesser side be subtracted, the remainder is 5; but if the greater side be added to 4 times the lesser side the sum is 30. Find the sides of the parallelogram.

Let $x + y$ = greater side.

$x - y$ = shorter side.

the longer side = 10

the shorter side = 5.

15. A possesses a certain number of acres; and if this number be added to B's acres, the sum is 110; but if B's acres be multiplied by 80, and from this product you subtract the square of A's acres there remains 1400. How many acres had each?

Suppose $55 + x$ = number of A's acres

$55 - x$ = do. B's

A possessed 60

B 50.

16. Divide the number (10) into two such parts that their product, added to the sum of their squares, shall be = 76.

the parts are 6 and 4.



17. Find two numbers, such that twice the first, added to three times the second, shall $=60$; and twice the square of the first, added to 3 times the square of the second, shall $=840$.

Let x = greater number.

And y = less number.

Answer, 18 and 8.

18. There are 4 numbers in geometrical proportion; the sum of the two middle terms is $=18$, and the sum of the extremes is 27: find the numbers.

Answer, 3, 6, 12, 24.

19. There are 3 numbers in Geometrical Progression, their sum is 19, and the sum of their squares is 133: find the numbers.

Let x, y, z be the numbers; $\therefore xz=y^2$

$$x+y+z=19 \quad \therefore x+\sqrt{xz}+z=19$$

$$x^2+y^2+z^2=133 \text{ and } x^2+xz+z^2=133$$

$$\therefore x-\sqrt{xz}+z=7$$

$$x+\sqrt{xz}+z=19, \text{ \&c.}$$

The numbers are 4, 6, 9.

20. Find two numbers, such that their difference multiplied into the difference of their squares, shall be $=32$, and their sum multiplied into the sum of their squares shall be $=272$.

Let x and y be the numbers.

$$\therefore \overline{x-y} \cdot \overline{x^2-y^2}=32; \quad \therefore \overline{x-y}^3=\frac{32}{x+y}$$

$$\overline{x+y} \cdot \overline{x^2+y^2}=272 \quad \therefore \overline{x+y}^3=\frac{272}{x+y}$$

$$\text{Hence we have } \overline{x+y}^3=\frac{512}{x+y} \quad \therefore \overline{x+y}^3=512, \text{ \&c.}$$

The numbers are 3 and 5.

21. Find two numbers such that the greater, added to the square of the less, is $=19$; and the greater, multiplied into the square of the less, $=90$.

Let x = greater number

And $\sqrt{y}$ = the less.

The numbers are 10 and 3.

22. Find two numbers such that their difference, added to the difference of their squares, may be $=14$; and their sum added to the sum of their squares may be $=26$.

The numbers are 4 and 2.

23. What two numbers are those whose sum is $=$ to their product; and their sum added to the sum of their squares is $=12$.

Let x and y be the numbers.

$x+y=xy; x^2+y^2+x+y=12; \therefore (x+y)^2=12+x+y, \&c$

The numbers are 2 and 2.

24. What two numbers are those whose sum, added to their product, is $=34$; and whose sum subtracted from the sum of their squares, leaves a remainder 42.

Let x = sum of the two numbers.

$\therefore 34-x$ = product of the numbers.

$42+x$ = sum of their squares.

And $42+x+2(34+x)=x^2$; &c.

The numbers are 4 and 6.

25. How many pounds have A and B, supposing that if the number of A's pounds be divided by B's, and the quotient be added to B's pounds, the amount is $=11$; and if the sum of both be multiplied by 4, the amount is $=22$.

The numbers are 11 and 44.

26. If the greater of two numbers be divided by 5, and added to the less, this sum is $=12$; and the product of the two numbers, divided by 4, is $=40$: find the numbers.





INVOLUTION.

LEMMA I.

I F $A+Bx+Cx^2 + \&c.$ be equal to $a+bx+cx^2 + \&c.$ for every possible value of x ;

Then $A = a$, $B = b$, $C = c$, &c.

F or if the equation be true independently of the value of x ,

let $x=0$ then every term involving $x = 0$, and $A = a$;

∴ the remainder $Bx+Cx^2 + \&c. = bx+cx^2 + \&c.$

or $B+Cx+Dx^2, \&c. = b+cx+dx^2, \&c.$

Then, by proceeding as before, $B = b$;

In like manner $C = c$, $D = d$, &c.

LEMMA II.

The value of the quantity $\frac{a^n-b^n}{a-b}$ (when b becomes = a) is na^{n-1} .

For by division we know that $\frac{a^n-b^n}{a-b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \&c. \dots$ to (n) terms; and when $b=a$, this expression becomes $a^{n-1} + a^{n-1} + a^{n-1} \dots$ to (n) terms, or na^{n-1} .

TO INVESTIGATE THE BINOMIAL THEOREM.

Take the binomial $\overline{a+b}^{\frac{m}{n}}$; now $a+b=a \times (1+\frac{b}{a})$

$$\therefore \overline{a+b}^{\frac{m}{n}} = a^{\frac{m}{n}} \times \overline{1+\frac{b}{a}}^{\frac{m}{n}}; \text{ let } \frac{b}{a}=x$$

Suppose that $1+x=v$

And that $1+y=z$

$$\therefore \underline{x-y=v-z}$$

$$\text{Assume } \overline{1+x}^{\frac{m}{n}} = 1 + Ax + Bx^2 + Cx^3 + \&c. = v^{\frac{m}{n}}$$

$$\text{Then will } \overline{1+y}^{\frac{m}{n}} = 1 + Ay + By^2 + Cy^3 + \&c. = z^{\frac{m}{n}}$$

$\therefore$ by subtraction

$$A \cdot \overline{x-y} + B \cdot (x^2-y^2) + C \cdot (x^3-y^3) + \&c. = v^{\frac{m}{n}} -$$

Divide by $x-y = v-z$

$$\begin{aligned} \text{And } A+B \cdot \frac{x^2-y^2}{x-y} + C \cdot \frac{x^3-y^3}{x-y} + D \cdot \frac{x^4-y^4}{x-y} + \&c. &= \frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v-z} \\ \text{or } A+B \cdot (x+y) + C \cdot (x^2+xy+y^2) + D \cdot (x^3+x^2y+xy^2+ & \\ \&c. &= \frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v-z} \end{aligned}$$

Now let $y=x$, $\therefore z=v$; and by Lem. 2.

$$A+2Bx+3Cx^2+4Dx^3+\&c. = \frac{m}{n} \times v^{\frac{m}{n}-1}$$

Multiply by $1+x=v$

$$\text{And we have } \left\{ \begin{array}{l} A+2Bx+3Cx^2+4Dx^3+\&c. \\ Ax+2Bx^2+3Cx^3+\&c. \end{array} \right.$$

$$\therefore A+\overline{A+2B}x+\overline{2B+3C}x^2+\overline{3C+4D}x^3+\&c. = \frac{m}{n}$$

$$\text{But } \frac{m}{n} \times v^{\frac{m}{n}} = \frac{m}{n} + \frac{m}{n}Ax + \frac{m}{n}Bx^2 + \frac{m}{n}Cx^3 + \&c.$$

$\therefore$ by Lemma 1. $A = \frac{m}{n}$; also,

$$A+2B=\frac{m}{n}A \quad \therefore 2B = \left(\frac{m}{n}-1\right)A = \frac{m}{n} \cdot \left(\frac{m}{n}-1\right).$$

In like manner D may be proved = $\frac{m}{n} \cdot \left(\frac{m}{n}-1\right) \cdot \left(\frac{m}{n}-2\right) \cdot \left(\frac{m}{n}-3\right) \cdot \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$

And in the same way the values of the remaining co-efficients $E, F, G, \&c.$ may be found. Hence it appears that

$$\frac{m}{n} \cdot \left(\frac{m}{n}-1\right) \cdot \left(\frac{m}{n}-2\right) \cdot \left(\frac{m}{n}-3\right) \cdot \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot x^4 + \&c.$$

And $\overline{a+b}^{\frac{m}{n}} = a \times 1 + \frac{b}{a} = \left(\text{substituting } \frac{b}{a} \text{ for } x\right)$

$$\frac{m}{n} \cdot \left(\frac{m}{n}-1\right) \cdot \frac{b^2}{a^2} + \frac{b^3}{a^3} + \frac{m}{n} \cdot \left(\frac{m}{n}-1\right) \cdot \left(\frac{m}{n}-2\right) \cdot \frac{b^3}{a^3} + \frac{b^4}{a^4} + \&c.$$

$$\frac{m}{n} \cdot \left(\frac{m}{n}-1\right) \cdot \frac{b^2}{a^2} + \frac{b^3}{a^3} + \frac{m}{n} \cdot \left(\frac{m}{n}-1\right) \cdot \left(\frac{m}{n}-2\right) \cdot \frac{b^3}{a^3} + \frac{b^4}{a^4} + \&c.$$

Cor. 1. If $n=1$; $\overline{a+b}^m = a^m + ma^{m-1}b + m \cdot \frac{m-1}{2} \cdot a^{m-2}b^2 + \frac{m-2}{3} a^{m-3}b^3 + \&c.$

Cor. 2. If for (b) we write $(-c)$; we have $\overline{a-c}^m = a^m - ma^{m-1}c + m \cdot \frac{m-1}{2} \cdot a^{m-2}c^2 - m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} a^{m-3}c^3 +$

&c. where each alternate term is negative.

Cor. 3. If a and b be each made $=1$, $\overline{a+b}^m = 2^m = 1 + m + m \cdot \frac{m-1}{2} + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} + \&c.$ which terms

are the co-efficients of the series $\overline{a+b}^m$.

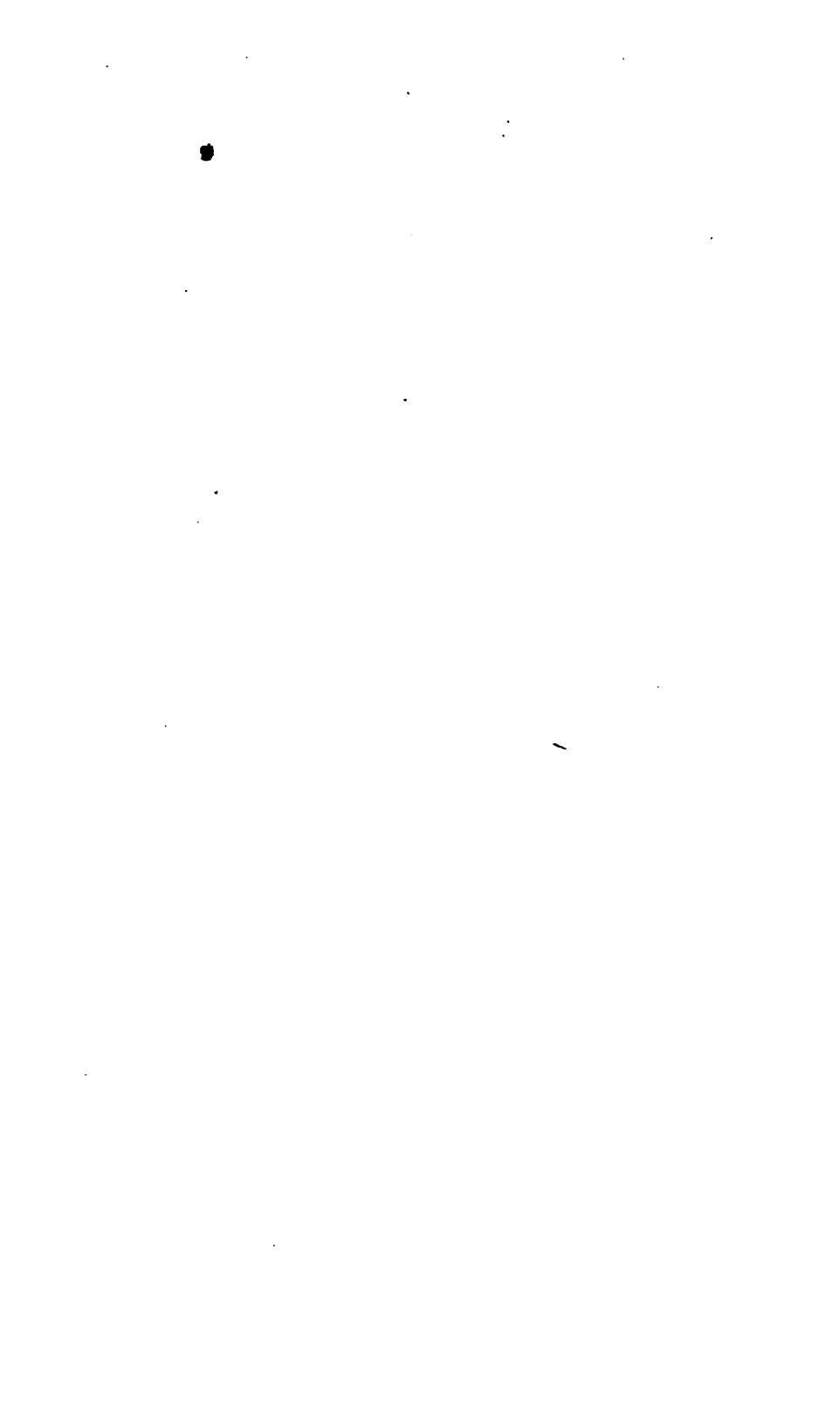
Cor. 4. To find the p^{th} term of $\overline{a+b}^m$. Since the 5th term is $m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot a^{m-4}b^4$, or $m \cdot \frac{m-1}{2}$

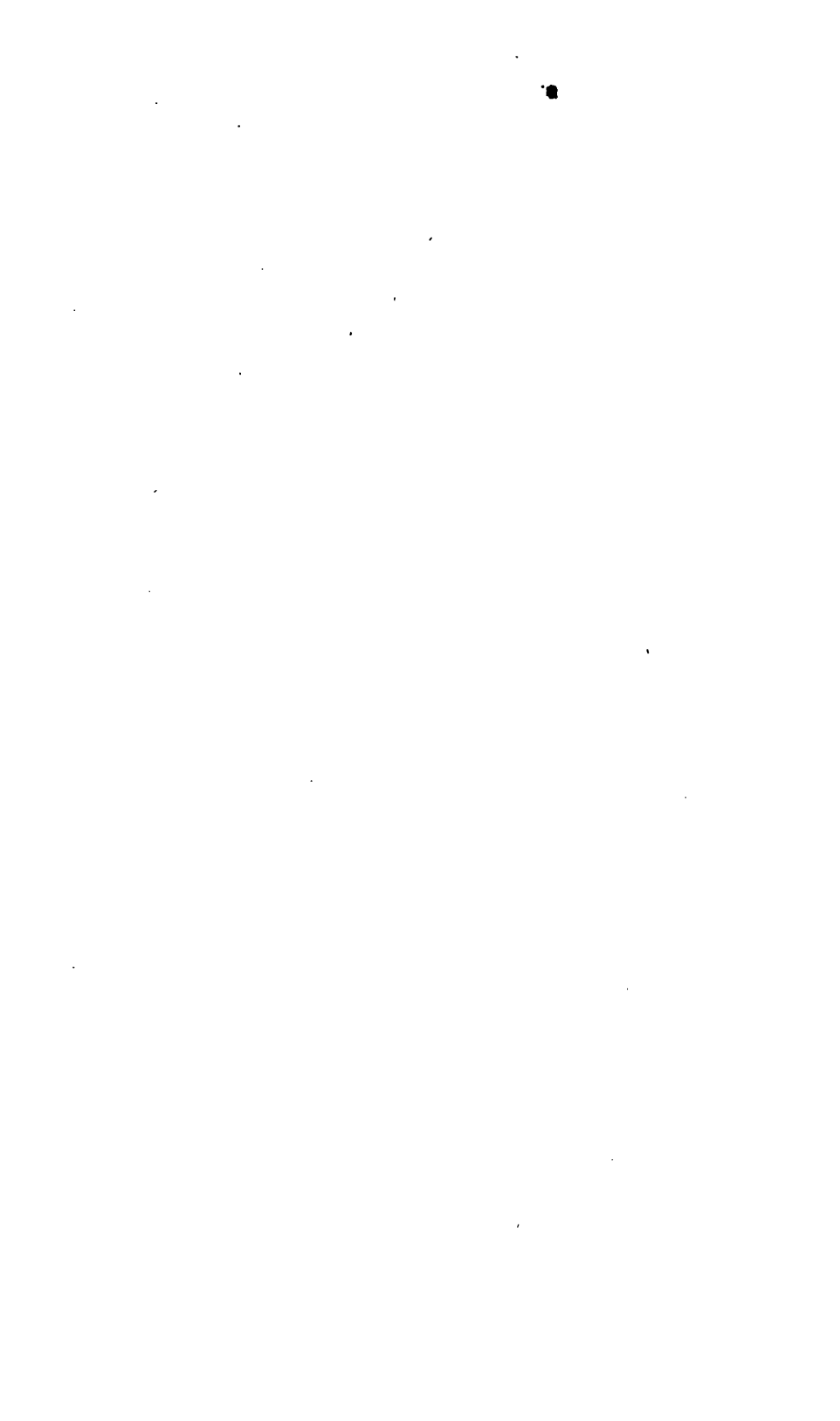
$\cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot a^{m-4}b^4$; the p^{th} term will be $m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \dots \cdot \frac{m-p+2}{p-1} \cdot a^{m-p+1}b^{p-1}$

Hence the $m+1^{\text{th}}$ term will be $m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot \dots \cdot \frac{m-m-3}{m-2} \cdot \frac{m-m-2}{m-1} \cdot \frac{m-m-1}{m} a^{m-m}b^m$

$= m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \dots \cdot \frac{3}{m-2} \cdot \frac{2}{m-1} \cdot \frac{1}{m} b^m = b^m$. If you proceed to find the $(m+2)^{\text{th}}$ term, the numerator of one fraction in the co-efficient becomes nothing, and consequently the term vanishes. The m^{th} term of the expanded binomial will be ma^mb^{m-1} . And by proceeding in the same manner we find the latter co-efficients of the series to be the same as those of the corresponding terms, taken from the beginning. This is manifest from ob-

... 1 ... 6 ... m-9





And also that $\overline{b+a}^m = b^m + mb^{m-1}a + m \cdot \frac{m-1}{2}b^{m-2}a^2 + \&c.$; which two series can only differ from each other by having their terms placed in opposite order; while the co-efficients of the corresponding terms are precisely the same.

Thus it appears that if (m) be a positive integer, the expanded binomial $\overline{a+b}^m$ consists of $\overline{m+1}$ terms. But if (m) be either fractional or negative, no term of the series can vanish; the series will therefore be continued in infinitum.

$$\text{Since } \overline{a+b}^m = a^m + ma^{m-1}b + m \cdot \frac{m-1}{2}a^{m-2}b^2 + \&c.$$

$$\text{And } \overline{a-b}^m = a^m - ma^{m-1}b + m \cdot \frac{m-1}{2}a^{m-2}b^2 + \&c.$$

$$\frac{\overline{a+b}^m + \overline{a-b}^m}{2} = a^m + m \cdot \frac{m-1}{2}a^{m-2}b^2 + m \cdot \frac{m-2}{3} \cdot \frac{m-1}{2} \times \frac{m-3}{4}a^{m-4}b^4 + \&c.$$

The sum of the odd terms of the binomial $\overline{a+b}^m$.

$$\text{Also } \frac{\overline{a+b}^m - \overline{a-b}^m}{2} = ma^{m-1}b + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}a^{m-3}b^3 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot \frac{m-4}{5} + a^{m-5}b^5; \text{ the}$$

sum of the even terms of the binomial $\overline{a+b}^m$.

Hence if $a=b=1$, we have $\frac{2^m}{2}$ ($=2^{m-1}$) = the sum of the co-efficients of the odd terms = the sum of the co-efficients of the even terms of the expanded binomial.

If the index (m) be fractional, the expanded binomial becomes an infinite series. Thus $\overline{a+b}^{\frac{1}{2}} = a^{\frac{1}{2}} \times$

$$\left(1 + \frac{b}{a}\right)^{\frac{1}{2}} = a^{\frac{1}{2}} \times \overline{1+x}^{\frac{1}{2}}, \text{ (if } x = \frac{b}{a} \text{). Now } \overline{1+x}^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{3x^3}{2 \cdot 4 \cdot 6} - \frac{3 \cdot 5x^4}{2 \cdot 4 \cdot 6 \cdot 8} + \&c.$$

$$\therefore \overline{a+b}^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{b}{2a} - \frac{b^2}{2 \cdot 4a^2} + \frac{3b^3}{2 \cdot 4 \cdot 6a^3} - \frac{3 \cdot 5b^4}{2 \cdot 4 \cdot 6 \cdot 8a^4} + \&c.$$

By means of this theorem we can approximate to the roots of numbers. Thus to extract the cube root of 31

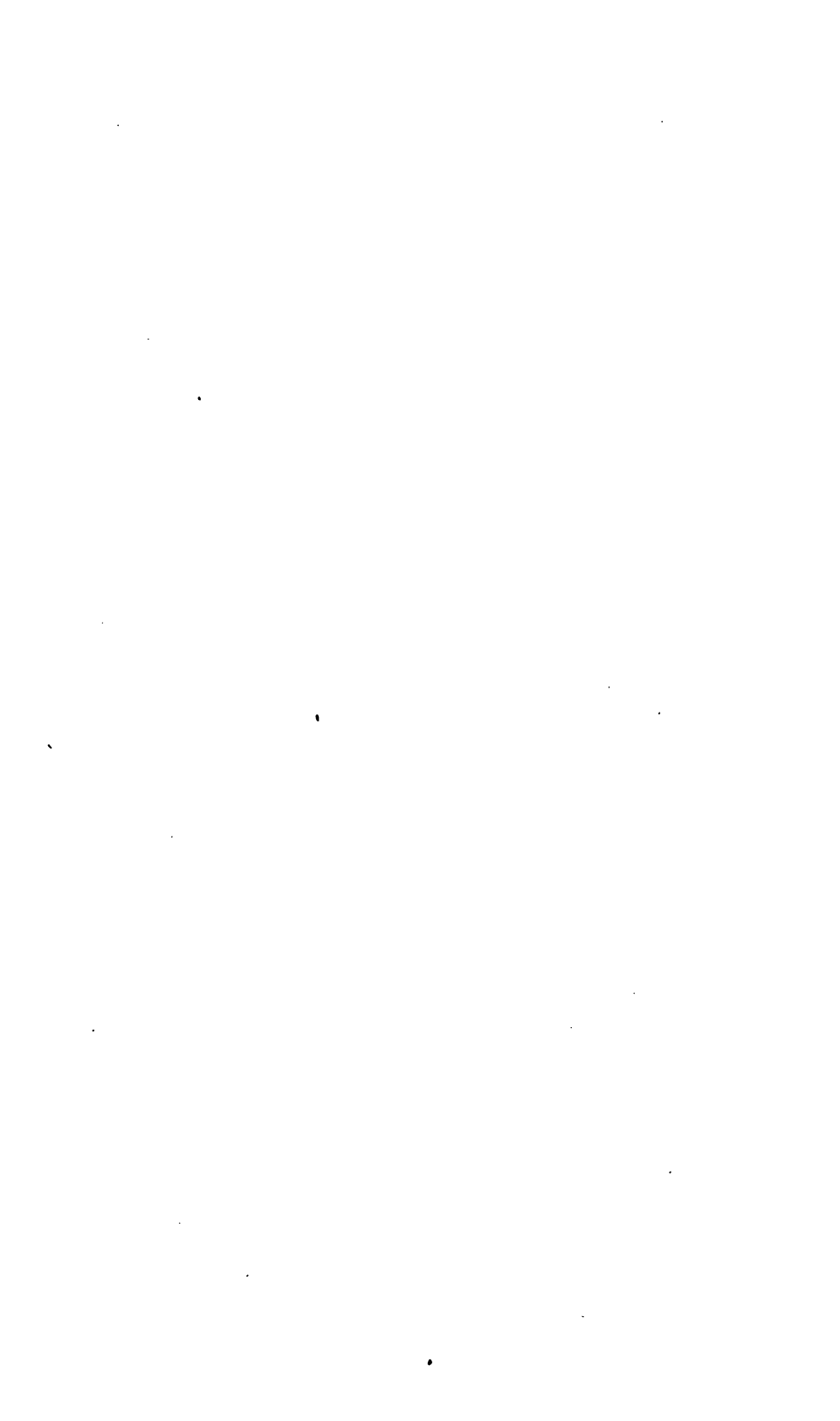
INVOLUTION.

$$\overline{a+b}^m = 31^{\frac{1}{3}} = \overline{27+4}^{\frac{1}{3}} = 27^{\frac{1}{3}} \times \overline{1+\frac{4}{27}}^{\frac{1}{3}} = 3 \times \overline{1+\frac{4}{27}}^{\frac{1}{3}} = 3 \times \left(1 + \frac{1}{3} \cdot \frac{4}{27} - \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{4^2}{729} + \frac{1}{3} \cdot \frac{4}{9} \cdot \frac{4^3}{19683} - \&c.\right)$$

$$\text{or } \sqrt[3]{27} = 3 \cdot \frac{4}{27} - \frac{16}{2 \cdot 187} + \frac{320}{321417} - \&c.$$

Suppose there are (n) sets of different quantities, and that $(a+b)$ represents the number in each set; all the possible collections that can be made out of the (n) sets (by taking one from each set) is $\overline{a+b}^n$; for *every one* of the $(a+b)$ quantities in the first set may be joined successively to *every one* of the $(a+b)$ quantities in the second set, and the number of *pairs* will be $\overline{a+b} \times \overline{a+b}$, or $\overline{a+b}^2$. In like manner each of these pairs may successively be joined with each of the $(a+b)$ quantities in the third set, and thus form $\overline{a+b}^3$ collections of three quantities. By proceeding in this manner with all the remaining sets, we shall have all the different collections that can be made out of the (n) sets (taking one from each set) = $\overline{a+b}^n$.





Suppose $a + b = 6$, the number of sides on a die

Let $a =$ one side (ace for instance); then $b = 5$

Take n (the number of dice) $= 2$

Then $\overline{a + b}^2 = 6^2 = 36$, all the different throws that can be made with two dice.

The index of (a) in any term, and the remaining quantity in the same term, respectively shew the number of aces and the different throws in which that number may occur. Thus the co-efficient of the first term is 1, and the index of (a) is 2; thence we infer that there is but one throw by which the two aces will be turned up. In the second term the index of (a) is 1, and $2b = 10$; there are therefore 10 different throws in which one ace only may appear. The third term, $b^2 (= 25)$ denotes the number of throws in which no ace appears. Upon this principle is founded the doctrine of Chances. Thus in the case before us, there are altogether $\overline{a + b}^2, (= \overline{1 + 5})^2$ or 36, different ways in which two dice may be thrown. It is but in one of these ways that the two aces may be thrown. There are ten different throws in which *one ace only* may be turned up, and there are 25 different throws of the two dice in which no ace may appear. Hence out of 36 throws the chances against an ace being thrown are 25; the chances of *one ace only* is 10; the chance of both aces is 1; the chance of *at least one* ace is 11. If we take four dice, $n = 4$; $\overline{a + b}^4 = 6^4 = 1296$, all the different ways in which four dice may be thrown. Now $\overline{a + b}^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$; the last term $b^4 (= 625)$ shews the number of different throws with four dice in which no ace may appear; the preceding term $4b^3 \times a = 500a$ denotes that there are 500 different throws by which *one ace* could be turned up; the third term $6b^2 \times a^2 = 150a^2$ shews that there are 150 different throws by which two aces may be turned up; the second term $4b \times a^3 = 20a^3$ shews that there are 20 different throws by which 3 aces may be turned up; and the first term a^4 shews that there is

but one throw by which the four aces may be turned 1
In all these cases the calculations are the same whether
suppose (m) dice to be thrown once, or one die to
thrown (m) times.

EVOLUTION.

THE method by which we extracted the cube root of .
gebraic Quantities may be applied to Roots in general.

Thus to extract the 5th root of $a^5x^5 - 5a^4x^4y + 10a^3x^3y^2 - 10a^2x^2y^3 + 5axy^4 - y^5$.

$$a^5x^5 - 5a^4x^4y + 10a^3x^3y^2 - 10a^2x^2y^3 + 5axy^4 - y^5 \quad (ax - y$$

$$5a^4x^4) - 5a^4x^4y$$

Place the 5th root of the first term in the quotient ; su
tract its 5th power from the proposed quantity ; divide
first term ($-5a^4x^4y$) of the remainder by 5 times the
power of the root (a), and place the quotient the secc
term in the root. Subtract the 5th power of the root th
found from the proposed quantity, and as there is no
mainder $ax - y$ is the root required.

Extract the 6th root of $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$.

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64 \quad (x -$$

$$6x^5) - 12x^5$$





Since the 6th power of $(x-2)$, when subtracted from the original quantity, leaves no remainder, $x-2$ is the root required. And in general, if it be required to find the n^{th} root of any proposed quantity, place (a) the n^{th} root of the first term in the quotient; subtract the n^{th} power of (a) from the proposed quantity; divide the first term of the remainder by na^{n-1} ; make the quotient the second term in the root; subtract the n^{th} power of the root thus found from the original quantity, and proceed as before till there be no remainder.

Before we proceed to extract the roots of surd quantities, the following propositions must be established.

1. If two quadratic surds, $\sqrt{a}$ and $\sqrt{b}$, cannot be reduced to others which have the same irrational part, their product is irrational.

For, if possible, let $\sqrt{a} \times \sqrt{b} = ma$

$$\therefore ab = m^2 a^2$$

$$\therefore b = am^2$$

$$\therefore \sqrt{b} = m\sqrt{a};$$

$\sqrt{b}$ is so reduced as to have the same irrational part as $\sqrt{a}$ has, which is contrary to hypothesis.

2. One quadratic surd, $\sqrt{x}$, cannot be made up of two others, $\sqrt{m}$ and $\sqrt{n}$, which have not the same irrational part.

For, if possible, let $\sqrt{x} = \sqrt{m} + \sqrt{n}$

$$\therefore x = m + 2\sqrt{mn} + n$$

$$\therefore \frac{x-m-n}{2} = \sqrt{mn}; \text{ but since } \sqrt{m}$$

and $\sqrt{n}$ have not the same irrational part, $\sqrt{mn}$ is irrational. Hence we find a rational quantity equal to an irrational one, which is impossible.

To extract the square root of the binomial $(a + \sqrt{b})$

where a is a rational quantity, and $\sqrt{b}$ is a quadratic surd.

$$\text{Let } \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$$

$$\therefore a + \sqrt{b} = x + 2\sqrt{xy} + y$$

then, by what was proved in Part. I., the rational parts both sides must be equal, and likewise the irrational

$$\text{Hence } a = x + y, \text{ and } \sqrt{b} = 2\sqrt{xy}$$

$$\therefore a^2 = x^2 + 2xy + y^2$$

$$b = 4xy$$

$$\therefore a^2 - b = x^2 - 2xy + y^2$$

$$\therefore \sqrt{a^2 - b} = x - y$$

$$a = x + y$$

$$\therefore \frac{a + \sqrt{a^2 - b}}{2} = x$$

$$\text{And } \frac{a - \sqrt{a^2 - b}}{2} = y$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

from which expression the root can always be found when $a^2 - b$ is a complete square.

Extract the square root of $11 + \sqrt{72}$

$$\text{Let } \sqrt{11 + \sqrt{72}} = \sqrt{x} + \sqrt{y}$$

$$\therefore 11 + \sqrt{72} = x + 2\sqrt{xy} + y$$

where $x + y = 11$, and $2\sqrt{xy} = \sqrt{72}$

$$\therefore x^2 + 2xy + y^2 = 121$$

$$4xy = 72$$

$$x^2 - 2xy + y^2 = 49$$

$$\therefore x - y = 7$$

$$x + y = 11$$

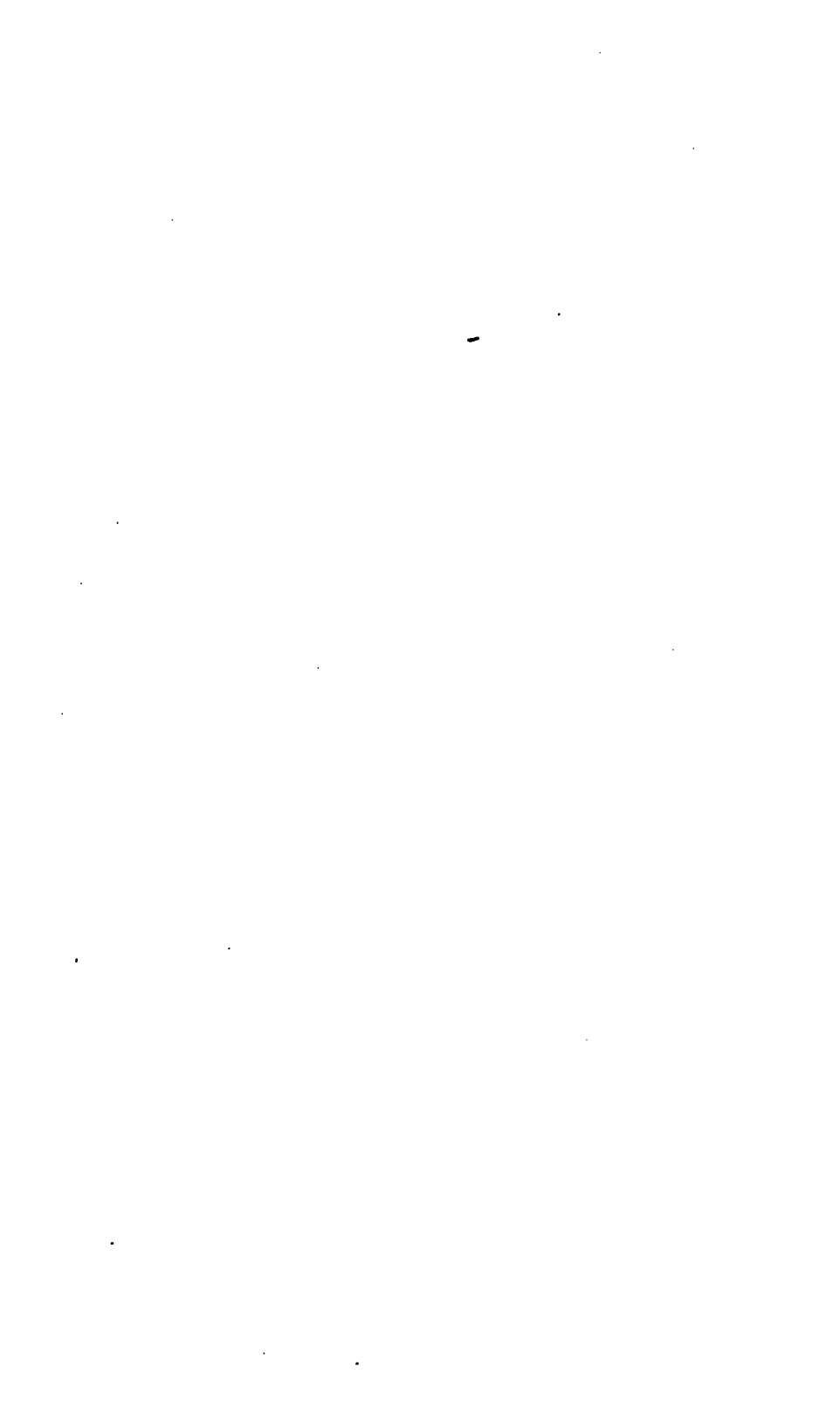
$$\therefore x = 9; y = 2;$$

$$\therefore \sqrt{x} + \sqrt{y} = 3 + \sqrt{2}.$$

That this is the true answer is manifest, for $(3 + \sqrt{2})^2 =$

$$9 + 2 = 11 + 6\sqrt{2} = 11 + \sqrt{72}.$$





Extract the square root of $6-2\sqrt{5}$

$$\text{Let } \sqrt{6-2\sqrt{5}} = \sqrt{x}-\sqrt{y}$$

$$\therefore 6-2\sqrt{5} = x-2\sqrt{xy}+y$$

$\therefore x+y=6$, and $2\sqrt{xy} = 2\sqrt{5}$; and by proceeding as before we find $x=5$, and $y=1$. $\therefore \sqrt{x}-\sqrt{y}=\sqrt{5}-1$.

This may be proved as before. Thus $(\sqrt{5}-1)^2 = 5-2\sqrt{5}+1 = 6-2\sqrt{5}$.

Extract the square root of $10-4\sqrt{6}$

$$\text{Let } \sqrt{10-4\sqrt{6}} = \sqrt{x}-\sqrt{y}$$

$$\therefore 10-4\sqrt{6} = x-2\sqrt{xy}+y.$$

$\therefore x+y=10$, and $2\sqrt{6} = \sqrt{xy}$; from these equations we find $x=6$, and $y=4$; $\therefore$ the root required is $\sqrt{6}-2$.

Extract the square root of $4\sqrt{-5}-1$.

$$\text{Let } \sqrt{-1+4\sqrt{-5}} = \sqrt{x}+\sqrt{y}$$

$$\therefore -1+4\sqrt{-5} = x+2\sqrt{xy}+y$$

$$\therefore x+y=-1; \quad \sqrt{xy}=2\sqrt{-5}$$

$$(x+y)^2=1 \quad \therefore xy=-20$$

$$4xy = -80$$

$(x-y)^2 = 81$; from which we find $x=4$ and $y=-5$ $\therefore \sqrt{x}+\sqrt{y}=2+\sqrt{-5}$, the true answer; for $(2+\sqrt{-5})^2 = 4+4\sqrt{-5}-5 = 4\sqrt{-5}-1$.

The same conclusions would be obtained from substituting for (a) and (b) in the general expression

$$\sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

1. Extract the square root of $7+2\sqrt{10}$.

Answer, $\sqrt{5}+\sqrt{2}$.

2. Extract the square root of $23-8\sqrt{7}$.

Answer, $4-\sqrt{7}$.

3. Extract the square root of $7-2\sqrt{10}$.

Answer, $\sqrt{5}-\sqrt{2}$.

4. Extract the square root of $18-10\sqrt{-7}$.

Answer, $5-\sqrt{-7}$.

5. Extract the square root of $6+2\sqrt{5}$.

Answer, $1+\sqrt{5}$.

By a similar process, the cube root of $a+\sqrt{b}$ may be found; where a is rational, and $\sqrt{b}$ is a quadratic surd.

$$\text{Let } \sqrt[3]{a+\sqrt{b}} = x+\sqrt{y}$$

$$\therefore a+\sqrt{b} = x^3+3x^2\sqrt{y}+3xy+y^{\frac{3}{2}}$$

$$\therefore a = x^3+3xy \text{ and } \sqrt{b} = 3x^2\sqrt{y}+y^{\frac{3}{2}}$$

$$\therefore a-\sqrt{b} = x^3-3x^2\sqrt{y}+3xy-y^{\frac{3}{2}}$$

$$\therefore \sqrt[3]{a-\sqrt{b}} = x-\sqrt{y}$$

$$\text{And since } \sqrt[3]{a+\sqrt{b}} = x+\sqrt{y}$$

$$\therefore \sqrt[3]{a^2-b} = x^2-y; \text{ Let } x^2-y=m \therefore y=x^2-m$$

$\therefore a(=x^3+3xy) = x^3+3x(x^2-m)$; from which equation the rational value of (x) may be found by trial.

Find the cube root of $10+\sqrt{108}$.

$$\text{Let } \sqrt[3]{10+\sqrt{108}} = x+\sqrt{y}$$

$$\therefore 10+\sqrt{108} = x^3+3x^2\sqrt{y}+3xy+y^{\frac{3}{2}}$$

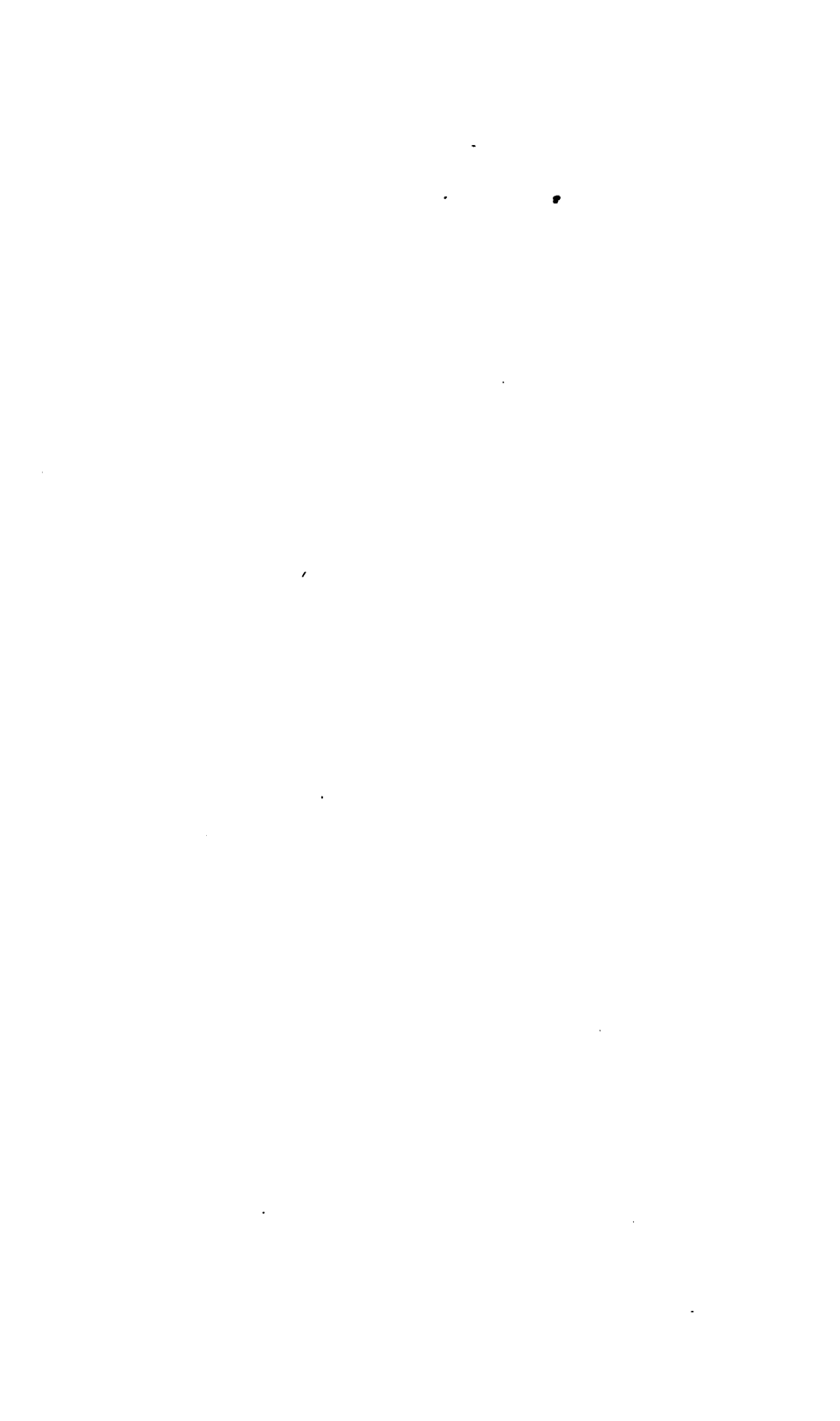
Then $x^3+3xy = 10$; and $\sqrt{108} = 3x^2\sqrt{y}+y^{\frac{3}{2}}$

$$\therefore \sqrt{108} - 10 = y^{\frac{3}{2}} - 3yx + 3x\sqrt{y} - x^3$$

$$\therefore \sqrt[3]{\sqrt{108}-10} = \sqrt{y}-x$$

$$\text{And } \sqrt[3]{\sqrt{108}+10} = \sqrt{y}+x$$





$$\therefore \sqrt[3]{108-100} = y - x^3$$

$$\text{or } 2 = y - x^3; \therefore y = x^3 + 2$$

$$\text{And } x^3 + 3xy = x^3 + 3x(x^3 + 2) = 10$$

$$\text{or } 4x^3 + 6x = 10; \text{ in which equation}$$

$$x = 1; \text{ and } y = 3;$$

$$\therefore x + \sqrt{y} = 1 + \sqrt{3}, \text{ the cube root required.}$$

Extract the cube root of $10 - 6\sqrt{3}$

$$\text{Let } \sqrt[3]{10-6\sqrt{3}} = x - \sqrt{y}; \text{ and by proceeding}$$

$$\text{before we find } x - \sqrt{y} = 1 - \sqrt{3}.$$

these solutions $a^3 \sqrt{b}$ is supposed to be a complete cube.

If it be required to extract the cube root of $\sqrt{a} - \sqrt{b}$ when both terms of the binomial are quadratic surds.

Let $\sqrt[3]{\sqrt{a} + \sqrt{b}} = x + y$, where x and y respectively involve the same surds that a and b do;

$$\text{Then } \sqrt{a} + \sqrt{b} = x^3 + 3x^2y + 3xy^2 + y^3$$

And since one quadratic surd cannot be made up of two others which have not the same irrational part, $\sqrt{a}$ must equal to the terms involving the same surd as it does,

$$\therefore \sqrt{a} = x^3 + 3xy^2.$$

$$\text{And } \sqrt{b} = y^3 + 3x^2y$$

$$\therefore \sqrt{a} - \sqrt{b} = x^3 - 3x^2y + 3xy^2 - y^3$$

$$\therefore \sqrt[3]{\sqrt{a} - \sqrt{b}} = x - y$$

$$\text{and } \sqrt[3]{\sqrt{a} + \sqrt{b}} = x + y$$

$$\therefore \sqrt[3]{\sqrt{a} - \sqrt{b}} = x - y$$

substitute (as before) for x , and its value will be obtained.

GREATEST COMMON MEASURE

AND LEAST COMMON MULTIPLE.

BEFORE we find the greatest common measure of two quantities, we may divide either of them by any quantity

which is not a measure of both, for by this means we evidently cannot affect the common measure of the two quantities. Thus the greatest common measure of (am) and (bm) is (m) ; and (m) is also the greatest common measure of (am) and (bcm) .

Suppose we have to find the greatest common measure of $x^4 - 6x^3 + 9x^2 - 4x$ and $x^5 - 2x^4 + 4x^3 - 6x^2 + 3x$; if we expunge (x) which divides both quantities, we must multiply it afterwards into the greatest common measure.

$$\begin{array}{r} x^3 - 6x^2 + 9x - 4 \quad) \quad x^4 - 2x^3 + 4x^2 - 6x + 3 \quad (x + 4 \\ \underline{x^4 - 6x^3 + 9x^2 - 4x} \end{array}$$

$$4x^3 - 5x^2 - 2x + 3.$$

$$4x^3 - 24x^2 + 36x - 16$$

$$\underline{19x^2 - 38x + 19}$$

$$\text{or } 19 \times (x^2 - 2x + 1);$$

Here 19 may be expunged, because it divides only one of the quantities; and $x^2 - 2x + 1$, which is found to be the greatest common measure, must be multiplied by (x) , the quantity before expunged from both divisor and dividend.

1. Find the greatest common measure of $x^4 + a^3x^2 + a^4$ and $x^4 + ax^3 - a^2x - a^4$.

$$\text{Answer, } x^2 + ax + a^2.$$

2. Find the greatest common measure of $x^3 - 3x^2 + 7x - 21$ and $2x^4 + 19x^2 + 35$.

$$\text{Answer, } x^2 + 7.$$

3. Find the greatest common measure of $6a^2 - 7ax - 3x^2$ and $6a^2 + 11ax + 3x^2$.

$$\text{Answer, } 3a + x.$$

4. Find the greatest common measure of $7x^2 - 12x + 5$ and $2x^3 + x^2 - 8x + 5$.

$$\text{Answer, } x - 1.$$

$$\begin{array}{r} x^4 + a^2x^2 + a^4 \big/ x^4 + ax^3 - a^3x - a^4 \quad (1) \\ \underline{x^4 + a^2x^2 + a^4} \\ ax^3 - a^3x - 2a^4 \end{array} \quad (\div \text{ by } a)$$

$$\begin{array}{r} x^3 - ax^2 - a^2x - 2a^3 \big/ x^4 + a^2x^2 + a^4 \quad (x+1) \\ \underline{x^4 - ax^3 - a^2x^2 - 2a^3x} \\ + ax^3 + 2a^2x^2 + 2a^3x + a^4 \quad (\div \text{ by } \\ + x^3 + 2ax^2 + 2a^2x + a^3 \\ + x^3 - ax^2 - a^2x - 2a^3 \\ (\div \text{ by } 3a) \quad 3ax^2 + 3a^2x + 3a^3 \end{array}$$

$$\begin{array}{r} x^2 + ax + a^2 \big/ x^3 - ax^2 - a^2x - 2a^3 \quad (x-2a) \\ \underline{x^3 + ax^2 + a^2x} \\ - 2ax^2 - 2a^2x - 2a^3 \\ - 2ax^2 - 2a^2x - 2a^3 \end{array}$$



LEAST COMMON MULTIPLE.

Three ponies, A, B, and C, trot round a plain 7 miles in circumference; A goes at the rate of $7\frac{1}{2}$ miles an hour, B at the rate of $8\frac{1}{2}$ miles an hour, and C at $9\frac{1}{4}$ miles an hour: when will they *first* come together?

B trots 1 mile an hour more than A; $\therefore$ in 7 hours B will have gained upon A by the circumference of the plain, and then they *first* come together again. C trots $\frac{3}{4}$ of a mile an hour more than B; then $\frac{3}{4} : 7 :: 1 : \frac{28}{3}$ time they *first* come together. They will also be together at the end of $m \times \frac{28}{3}$ hours. But as A and B can come together only at the expiration of *every* 7 hours, A, B, and C can come together only at the expiration of some multiple of 7 hours. Let then $n \times 7 = m \times \frac{28}{3}$; now the *least possible* value of (m) which will make this latter quantity a whole number is 3; $\therefore n = 4$; and the *first time* A, B, and C come together is at the end of 28 hours.

A, B, and C start together to travel round an island 100 miles in circumference; A goes 15 miles, B goes 10 miles, and C 6 miles a-day. If they continue to travel till they all meet again, what time will have elapsed since they first started?

A travels 5 miles a-day faster than B; and the first time they come together again is when A has travelled round the island once more than B,

$$\therefore 5 : 100 :: 1 : \frac{100}{5} = 20, \text{ the first time A and B meet.}$$

In like manner

$$4 : 100 :: 1 : \frac{100}{4} = 25, \text{ the first time B and C meet;}$$

Then it is evident that A and B would come together every 20 days; also B and C would be together at the end of every 25 days. The *least* common multiple then of 20 and 25 is the *first* time they all come together again. Now since the greatest common measure of 20 and 25 is 5 the least common multiple = $\frac{20 \times 25}{5} = 100$.

In general, if

P = periodic time of A round a circumference (S)

and p = ... B

and p' = ... C

day.

day.

Then $P : 1 :: S : \frac{S}{P}$ = space described in 1 by A

$p : 1 :: S : \frac{S}{p}$ = ditto by B

$p' : 1 :: S : \frac{S}{p'}$ = ditto by C

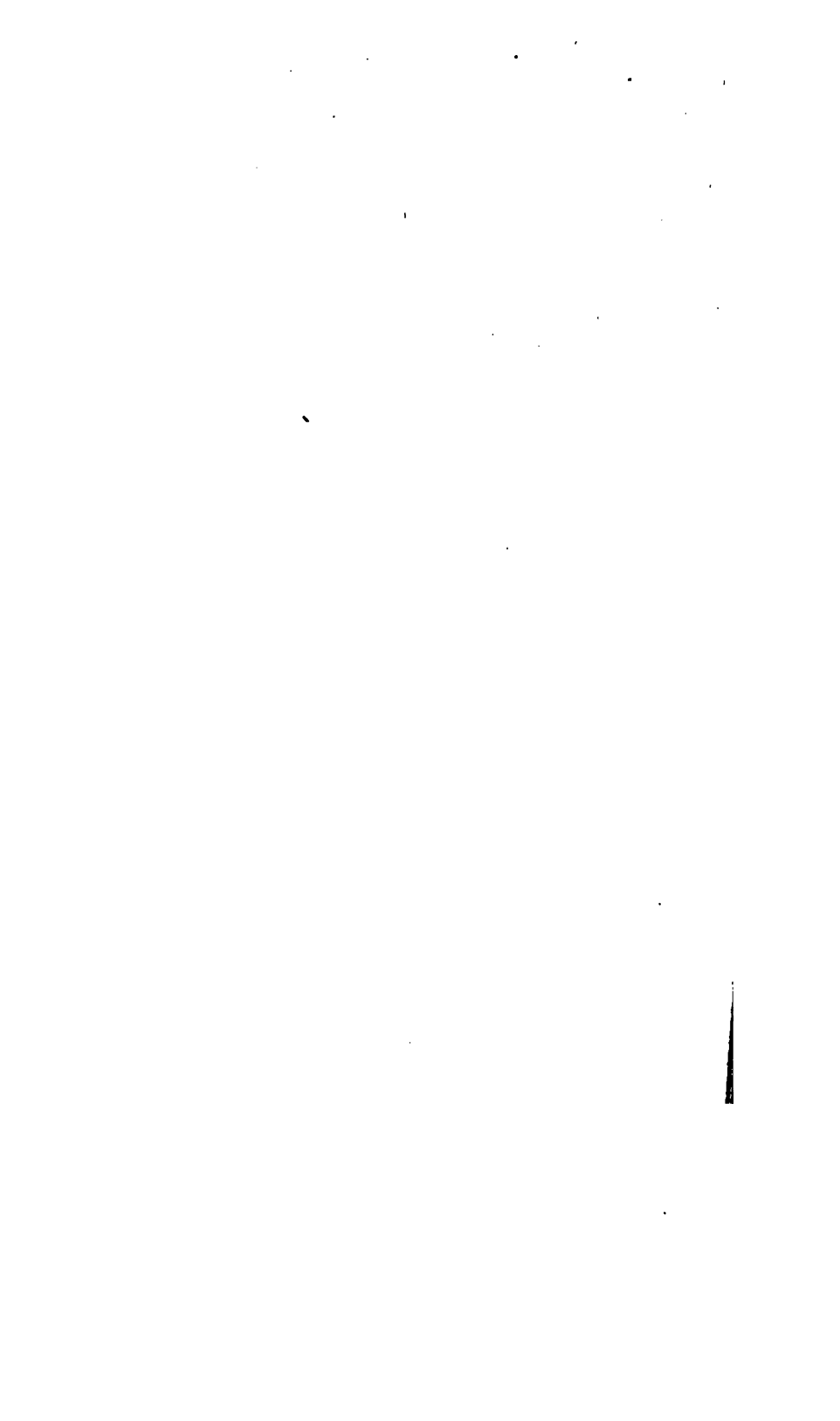
$\therefore \frac{S}{p} - \frac{S}{P} = \frac{P-p}{Pp} \cdot S$ = excess of B 's travel above A in 1 day

$\frac{S}{p} - \frac{S}{p'} = \frac{p'-p}{pp'} \cdot S$ = do. of C above B in do.

Also $\frac{P-p}{Pp} S : S : 1 : \frac{Pp}{P-p}$, time in which gains upon A by the whole circumference.

In like manner $\frac{pp'}{p'-p} =$ time in which C gains upon B by the whole circumference.

The *least common multiple* of the two times will be the *first time* the three bodies come together.





There is an island 29 miles in circumference; three travellers, A, B, and C, start together to travel round it; A goes 3 miles an hour, B 5, and C 7: when will they all come together again?

B gains 2 miles an hour upon A;

$\therefore 2 : 29 :: 1 : \frac{29}{2}$, the time A and B first come together.

C gains 4 miles an hour upon A;

$\therefore 4 : 29 :: 1 : \frac{29}{4}$, the time A and C first come together.

A and C will also come together in $2 \times \frac{29}{4}$ (or $\frac{29}{2}$) hours; hence in $14\frac{1}{2}$ hours, A, B, and C will first come together.

We should evidently arrive at the same conclusion if we found the least common multiple of the two times, according to rule: Thus $\frac{29}{2} \times \frac{29}{4} = \frac{29 \times 29}{8}$; and the greatest common measure of the two quantities is $\frac{29}{4}$; therefore the least common multiple is $\frac{29 \times 29}{8} \times \frac{4}{29} = \frac{29}{2}$.

A, B, C, and D are the hands of a clock: A goes round in 5 days and 20 hours; B in 7 days and 14 hours; C in 10 days and 20 hours; and D in 18 days and 23 hours; after what interval of time will all the hands be together?

	hours;		hour.
A revolves in 140 and $\therefore$ describes $\frac{1}{140}$ of a revolution in 1			
B	182.....	$\frac{1}{182}$	
C	260.....	$\frac{1}{260}$	
D	455.....	$\frac{1}{455}$	

Since the hands are supposed to start together, it is manifest that the *least common multiple* of those four numbers is the *first time* they come together *at the same place*: the least common multiple is 1820; $\therefore$ in 1820 hours the four hands will be together again *at the point from which they started*. But to find the *first time* they all come together, we find, (as in preceding examples,)

$\frac{1}{140} - \frac{1}{435} = \frac{9}{1820}$ of the circumference which A gains upon D in one hour ;

$\therefore \frac{9}{1820} : 1 :: 1 \text{ hour} : \frac{1820}{9} = 202\frac{2}{9}$, the time when A and D first come together.

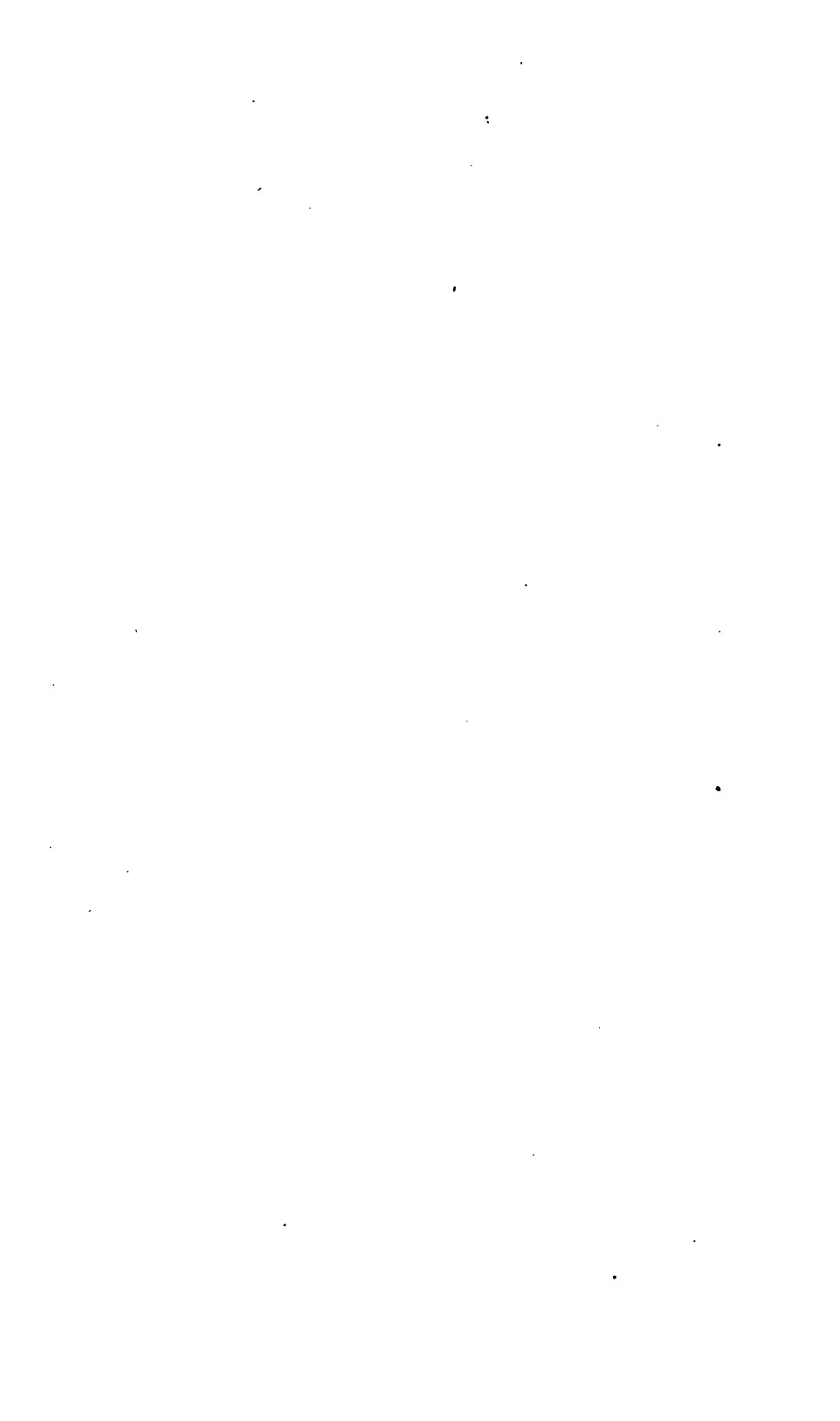
Again $\frac{1}{182} - \frac{1}{435} = \frac{3}{910}$ of the circumference which B gains upon D in one hour ;

$\therefore \frac{3}{910} : 1 :: 1 \text{ hour} : \frac{910}{3} = 303\frac{1}{3}$, the time when B and D first come together.

In like manner $606\frac{2}{3}$ hours will be the time when C and D first come together.

And since $606\frac{2}{3}$ is a multiple of $202\frac{2}{9}$ and $303\frac{1}{3}$, the first time the four hands come together is at an interval of $606\frac{2}{3}$ hours.





PERMUTATIONS AND COMBINATIONS.

IF R denote the number of permutations that can be formed out of $\overline{n-1}$ quantities, taken together; nR is the number of permutations that can be formed out of (n) quantities, taken together. For the additional quantity may be placed *either before or after each* of the other $(n-1)$ quantities in every permutation, and thus have (n) *different places* in each of the (R) permutations. Hence there will be (n) permutations formed out of each of the original (R) permutations; $\therefore$ upon the whole there must be nR permutations.

Let then Q denote the number of permutations that can be formed out of (n) quantities taken together,

R ditto out of $\overline{n-1}$ quantities taken together,

S ditto out of $\overline{n-2}$ quantities

T ditto out of $\overline{n-3}$ quantities

then it appears that $Q = nR$,

In like manner $R = \overline{n-1} . S$,

And $S = \overline{n-2} . T$, and so on till we arrive at $(\overline{n-n-1})$, or 1 quantity, which admits of one permutation. Hence Q , the number of permutations which (n) quantities admit, taken together $= nR = n . \overline{n-1} . S = n . \overline{n-1} . \overline{n-2} . T = n . \overline{n-1} . \overline{n-2} . \overline{n-3} \dots 2 . 1$.

In like manner if it be required to find the number of permutations that can be formed out of (n) quantities taken (r) together;

Let P denote the number of permutations that can be formed out of (n) quantities taken (r) together.

Q ditto, out of $(n-1)$ quantities taken $(r-1)$ together

R ditto, out of $(n-2)$ $(r-2)$
 &c. &c.

W ditto, out of $(n-r-1)$ $(r-r-1)$

Now if we add a new quantity (p) to the $(n-1)$, and take (r) of them together, we shall have (Q) permutations when (p) stands first. We shall also have (Q) permutations when *each* of the (n) quantities stands first: therefore, upon the whole, we shall have (nQ) permutations formed out of (n) quantities taken (r) together

Therefore $nQ = P$

In like manner $\overline{n-1} \cdot R = Q$

and $\overline{n-2} \cdot S = R$

&c. &c.

$$\therefore P = nQ = n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \dots \dots (n-r-1)$$

To find the number of permutations that can be made out of (n) quantities, taken all together, when there are (p) of one sort, (q) of another, and (r) of a third sort.

If all the quantities were different, the number of permutations would be $n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \dots 3 \cdot 2 \cdot 1$.

If the (p) quantities were different, they would form $1 \cdot 2 \cdot 3 \dots p$ permutations. If the (q) quantities were different they would form $1 \cdot 2 \cdot 3 \dots q$ permutations. If the (r) quantities were different they would form $1 \cdot 2 \cdot 3 \dots r$ permutations. But in the present case they each form only one permutation, therefore the whole number of permutations must be

$$\frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots 3 \cdot 2 \cdot 1}{(1 \cdot 2 \cdot 3 \dots p) \times (1 \cdot 2 \cdot 3 \dots q) \times (1 \cdot 2 \cdot 3 \dots r)}$$

To find the number (N) of COMBINATIONS that can be formed out of (n) quantities, taken (r) together.





Since every combination of (r) quantities, taken all together, admits $1 \cdot 2 \cdot 3 \dots (r)$ permutations; $\therefore$ the whole number of permutations is $(1 \cdot 2 \cdot 3 \dots r) N$; but the whole number of permutations is also $n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-r-1)}$. Therefore $(1 \cdot 2 \cdot 3 \dots r) N = n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-r-1)}$. $\therefore N = \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-r-1)}}{1 \cdot 2 \cdot 3 \dots r}$

$$\text{If } r=2, N = \frac{n \cdot \overline{n-1}}{1 \cdot 2}$$

$$\text{If } r=3, N = \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{1 \cdot 2 \cdot 3}$$

$$\text{If } r=4, N = \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$\text{If } r=n, N = \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{3} \cdot \overline{2} \cdot \overline{1}}{1 \cdot 2 \cdot 3 \dots n} = 1$$

Since N , the number of combinations, must be an integer, the quantity $\frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-r-1)}}{1 \cdot 2 \cdot 3 \dots r}$ must be an integer; *i. e.* the continued product of all the integers from (n) to $(n-r-1)$ inclusive, is divisible by the continued product of all the integers from 1 to r inclusive.

The number of combinations formed out of (n) quantities, taken $(n-r)$ together is the same as the number of combinations formed out of (n) quantities, taken (r) together.

Let $n-r=m$, $\therefore n-m=r$; then, by what has been shewn, the number of combinations of (n) quantities, taken (m) together, is

$$\frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-m-1)}}{1 \cdot 2 \cdot 3 \dots m} = \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots (r+1)}{1 \cdot 2 \cdot 3 \dots (n-r)}$$

or more fully expressed thus :

$$\frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-r-1)} \cdot \overline{n-r} \dots \overline{r+1}}{1 \cdot 2 \cdot 3 \cdot 4 \dots \overline{(n-r+3)} \cdot \overline{(n-r+2)} \cdot \overline{(n-r+1)} \cdot \overline{n-r}}$$

$$= \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{(n-r-1)}}{1 \cdot 2 \cdot 3 \dots r}, \text{ the same expression as}$$

we found for the number of combinations of (n) quantities taken (r) together. We assumed $(n-r)$ to be greater than $(r+1)$, and $\therefore$ as the terms in the numerator successively decrease, and those in the denominator successively increase by unity, all the terms from $\overline{n-r}$ to $\overline{r+1}$ inclusive, are cancelled, as they belong both to numerator and denominator. If $(n-r)$ be less than $(r+1)$, multiply both numerator and denominator by all the successive integers between $(n-r)$ and $(r+1)$, and the expression becomes

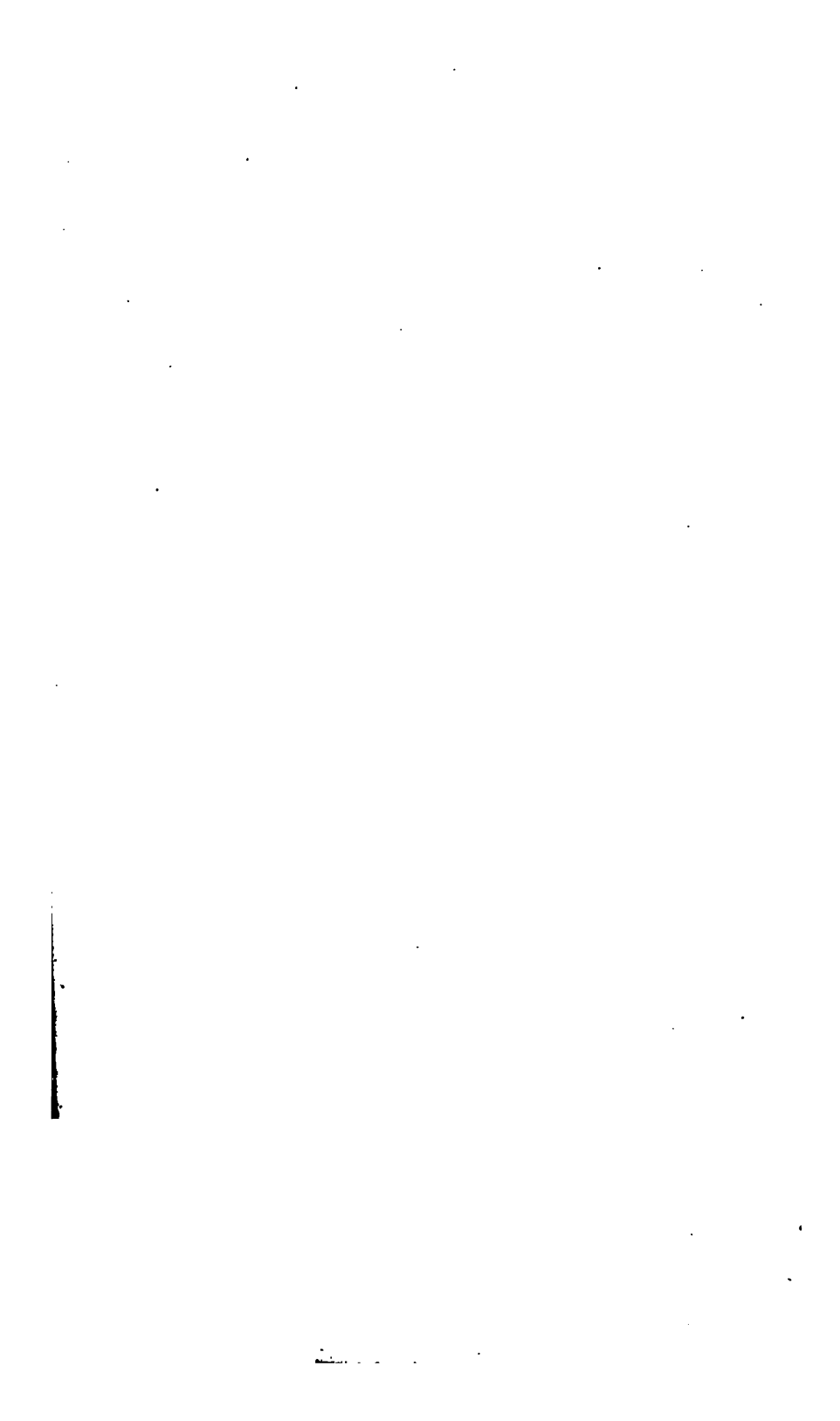
$$\frac{n \cdot \overline{n-1} \cdot \overline{n-2} \dots \overline{r+1} \cdot r \dots \overline{(n-r-1)}}{1 \cdot 2 \cdot 3 \dots n-r \cdot \overline{n-r-1} \cdot \overline{n-r-2} \dots r}$$

the same expression as before.

If there be (r) different sets of quantities, and the number of one set be N , the number of the second be n , and of the third set be ν , and so on to (r) sets; find the number of combinations that can be made out of the (r) sets, by taking one quantity at a time from each set.

Since each of the (N) quantities may be joined to each of the (n) quantities, there must be, upon the whole, $N \times n$ combinations arising from these two sets; and as these (Nn) combinations may be joined to each of the (ν) quantities in the third set, there will be $N \times n \times \nu$ combinations of three quantities arising from the three sets; and proceeding in this manner we shall have $N \times n \times \nu \times (\dots \text{to } r \text{ terms})$ combinations of (r) quantities arising from the (r) sets. Suppose, for example, we wish to know all the different ways in which three dice may be thrown: here $r=3$; $N=n=\nu=6$; and $N \times n \times \nu \dots r$ becomes $6 \times 6 \times 6 = 6^3 = 216$. If we take two dice, we have $6^2 = 36$; if 4 dice, we have $6^4 = 1296$. This result agrees with our conclusions derived from the expansion of the binomial $(a+b)^n$.





PROGRESSION.

ARITHMETICAL PROGRESSION.

A and B start together from the same place. A travels 9 miles the first day, 19 the second, 29 the third, and so on; B. travels 34 miles every day. When will A overtake B?

Let x = days required before A overtakes B.

B travels $34x$ miles.

A travels $9 + 19 + 29 + \dots + 9 + x - 1 \cdot 10$ miles; and

this series is $= (9 + 9 + x - 1 \cdot 10) \times \frac{x}{2}$

$$\therefore (9 + 9 + x - 1 \cdot 10) \times \frac{x}{2} = 34x$$

$$\therefore 9 + 9 + x - 1 \cdot 10 = 68$$

$$\therefore 18 + 10x - 10 = 68 \therefore 10x = 60 \therefore x = 6$$

A and B leave London and Plymouth at the same time, and travel till they meet. A travels 14 miles the first day, 18 the second, 22 the third, and so on; B travels 8 miles the first day, and increases his march daily by 4 miles. When will they meet, and what space will each have passed over, supposing Plymouth and London to be 217 miles apart?

Let x = number of days before they meet,

A travels $14 + 13 + 12 + \dots + 14 + 1 - x$ miles

$$= \overline{14 + 15 - x} \times \frac{x}{2} = \overline{29 - x} \times \frac{x}{2}$$

B travels $8 + 12 + 16 + \dots + 8 + \overline{x - 1} \cdot 4$ miles

$$= (8 + 8 + \overline{x - 1} \cdot 4) \times \frac{x}{2} = (12 + 4x) \times \frac{x}{2}$$

$$\therefore (12 + 4x + 29 - x) \cdot \frac{x}{2} = 217$$

$$\text{or } (41 + 3x) \cdot \frac{x}{2} = 217.$$

See Quadratic Equations. ^x

A buys 9 cakes; the prices of which are in arithmetical progression. The price of the cheapest but one was $10d.$ and the price of the dearest was $2s.$; find the price of each.

$\neq$ Let x = price of the cheapest,
 y = common difference,

The prices are, $x, x + y, x + 2y, \dots, x + 8y$

$$\text{But } x + 8y = 24$$

$$\text{And } x + y = 10$$

$$\therefore 7y = 14$$

$$\therefore y = 2$$

$$\text{and } x = 8$$

There are three numbers in arithmetical progression, whose common difference is 2, and sum = 21; find them.

Let $x - 2, x, x + 2$ be the numbers,

$$\therefore 3x = 21 \therefore x = 7.$$

There are four numbers in arithmetical progression, whose common difference is 4, and sum = 100; find them.

Let $x - 6, x - 2, x + 2, x + 6$ be the numbers

$$\therefore 4x = 100 \therefore x = 25.$$

$$x^2 + \frac{41}{3}x + \left(\frac{41}{6}\right)^2 = \frac{1601}{36} + \frac{434}{3} = \frac{6009}{36}$$

$$x + \frac{41}{6} = \pm \frac{83}{6}$$

$x = 7$ or $-20\frac{2}{3}$, the second answer is evidently improbable.

A travelled $29 - x + \frac{x}{2} = 22 + 7\frac{1}{2} = 77$

B travelled $12 + 4x \cdot \frac{x}{2} = 20 + 7\frac{1}{2} = 140$

$$77 + 140 = 217.$$

≠ otherwise. $n = 9$.

$$a + n - 1 \cdot b = 2$$

$$a + b = 1$$

$$n - 2 \cdot b =$$

$$b =$$

$$\therefore a = 8.$$

$$x-2y, x-y, x, x+y, x+2y$$

x-13

$$x^2 - 13x + \frac{13}{2} \Bigg)^2 = \frac{169}{4} - 12 = \frac{121}{4}$$

$$x - \frac{13}{2} = \pm \frac{11}{2}$$

$$x = 12 \text{ or } 1.$$

will overtake A after the first day the

There are five numbers in arithmetical progression, whose sum is 200; and the difference between the first and last is 20: find the numbers.

~~* Let $x-2y, xy, x, x+y, x+2y$ be the numbers,~~

$$\therefore 5x = 200 \therefore x = 40$$

$$\text{also } 4y = 20 \therefore y = 5$$

The numbers are 30, 35, 40, 45, 50.

Apples are placed in parallel rows of 1, 2, 3, 4, &c. so as to cover half of a square floor. If 400 apples be added to the number they will cover half a square floor which would contain one apple more on each of its sides. What is the number of apples at first?

Let x = the number of apples in the last row of the triangle.

The quantity of apples in the first instance is

$$1+2+3+\&c.\dots+x=\overline{1+x}\times\frac{x}{2}$$

$$\therefore \overline{1+x}\times\frac{x}{2}+400=\frac{1}{2}(x+1)^2$$

$$\therefore x+x^2+800=x^2+2x+1$$

$$\therefore x=799$$

And the number of apples at first is

$$\overline{1+x}\times\frac{x}{2}=\frac{800\times 799}{2}=799\times 400=319600$$

A travels 1 mile the first day, 2 the second, 3 the third, and so on; B starts 3 days after him, and travels 10 miles a day: in what time will B overtake A?

Let x = the number of days required,

$\therefore x+3$ = number of days A travels,

A travels $1+2+3+\dots+x+3$ miles,

$$=(x+3+1)\times\frac{x+3}{2}=\frac{x^2+7x+12}{2}$$

B travels $10x$ miles,

$$\therefore \frac{x^2+7x+12}{2}=10x$$

$$\therefore x^2+7x+12=20x$$

* $\therefore x^2-13x=-12$; from which equation (x) may be found.

The product of five numbers in arithmetical progression is 400, and their sum = 20; what are the numbers?

Let $x-2y, x-y, x, x+y, x+2y$ be the numbers

$$\therefore 5x = 20 \therefore x = 4$$

$$\therefore 4 \times (16 - 4y^2) \times (16 - y^2) = 400$$

$$\therefore (16 - 4y^2) \times (16 - y^2) = 100; \text{ from which}$$

equation the value of (y) may be found.

$$y = \sqrt{10 \pm \sqrt{61}} = 4.22 \text{ or } 1.479$$

Given the sum (S) and the sum of the squares (a) of five numbers in arithmetical progression; to find the numbers.

Let $x-3y, x-y, x, x+y, x+3y$ be the numbers.

$$\therefore 4x = S$$

$$(x-3y)^2 + (x-y)^2 + x^2 + (x+y)^2 + (x+3y)^2 = a$$

$$\text{Or } 4x^2 + 20y^2 = a$$

$$\therefore \frac{S^2}{4} + 20y^2 = a; \text{ \&c. } y = \sqrt{\frac{4a - S^2}{80}}.$$

The sum of five numbers in arithmetical progression 25; and their continued product is 945: find the numbers.

Solve the equation $x^5 - 6x^3 + 11x = 6$; in which the values of x are in arithmetical progression.

In every equation the co-efficient of the second term equal to the sum of the roots with their signs changed:

Suppose the values of x to be $a-b, a, a+b$, then $-3a = -6$, $\therefore a = 2$, one value of x .

Divide the equation $x^5 - 6x^3 + 11x - 6 = 0$ by $x-2$: and we obtain the quadratic $x^2 - 4x + 3 = 0$; in which two values of x are 1 and 3. Therefore the roots of proposed equation are 1, 2, and 3.

Find from the general expression the value of (n) in terms of S, a , and b .

$$S = \frac{2a + n - 1}{2} \times b$$

$$\therefore 2S = 2an + bn^2 - bn$$

* This answer is not quite correct but might be made more so by carrying the dec. further. The series (supposing $\eta = 4.22$) be 12.44, 8.22, 4., $-.22$, -4.44 . The product of which = 399.54 or very nearly

for an explanation of the negative
values of α see Proc. Alg. p. 677.

$$\therefore n^2 + \frac{2a-b}{b} \times n = \frac{2S}{b}$$

$$\therefore n^2 + \frac{2a-b}{b} \times n = \frac{2S}{b}$$

$$\therefore n^2 + \frac{2a-b}{b} \times n + \left[\frac{2a-b}{2b} \right]^2 = \left[\frac{2a-b}{2b} \right]^2 + \frac{2S}{b}$$

$$\therefore n + \frac{2a-b}{2b} = \pm \sqrt{\left[\frac{2a-b}{2b} \right]^2 + \frac{2S}{b}}$$

$$\therefore n = \pm \sqrt{\left[\frac{2a-b}{2b} \right]^2 + \frac{2S}{b}} - \frac{2a-b}{2b}$$

A traveller who has to pass over 192 miles, travels 28 miles the first day, 26 the next, 24 the next, and so on: In how many days will he reach his journey's end?

Let x = the number of days required,

$$a = 28$$

$$b = -2$$

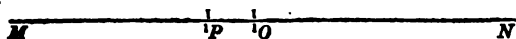
$$S = 192$$

By substituting for these quantities in the expression just

$$\begin{aligned} \text{found, we have } x &= \pm \sqrt{\frac{29^2}{4} - 192} + \frac{29}{2} \\ &= \pm \sqrt{\frac{841}{4} - 192} + \frac{29}{2} \\ &= \frac{\pm \sqrt{73} + 29}{2} = 10 \text{ } 5 \text{ } 28. \\ &\quad \text{or } = 18 \text{ } 10 \text{ } 31. \end{aligned}$$

GEOMETRICAL PROGRESSION.

A GEOMETRIC mean between two given quantities (a) and (b) is less than an Arithmetic mean between the same quantities.



Let $MP = a$, and $PN = b$; bisect MN in O ; if a semi-circle were described upon MN , and a perpendicular to

MN drawn through P meeting the circumference; perpendicular is the geometric mean, and is less than radius of the circle, which is the arithmetic mean, betw the two quantities. It may also be proved thus; let a and b be the two quantities as before, then the square $(a-b)$ must be a positive quantity.

or $a^2 - 2ab + b^2$ is positive

$\therefore a^2 + b^2$ is greater than $2ab$

$\therefore a^2 + 2ab + b^2$ is greater than $4ab$

$\therefore a + b$ is greater than $2\sqrt{ab}$

$\therefore \frac{a+b}{2}$ is greater than $\sqrt{ab}$

But $\frac{a+b}{2}$ is the arithmetic mean, and $\sqrt{ab}$ is the geometric mean, between (a) and (b) .

Solve the equation $x^3 - 7x^2 + 14x = 8$, where the values of (x) are in geometrical progression.

Suppose a, ar, ar^2 to be the 3 values of (x) in the equation $x^3 - 7x^2 + 14x - 8 = 0$,

Then $-a^3r^3 = -8 \therefore ar = 2$

Divide the equation by $(x-2)$; and we then have solve a quadratic, in which we find the other two values of (x) to be 1 and 4.

The sum of the first and third of four numbers in geometrical progression is 148; and the sum of the second and fourth is 888: find the numbers.

Let x, xy, xy^2, xy^3 be the numbers

Then $x + xy^2 = 148$

$xy + xy^3 = 888$

Now $xy + xy^3 = 148y = 888$

$\therefore y = \frac{888}{148} = 6$

Again $x(1 + y^2) = 148$

Or $x(1 + 36) = 148$

$\therefore x = \frac{148}{37} = 4$

Then the numbers are 4, 24, 144, and 864.

- direct method

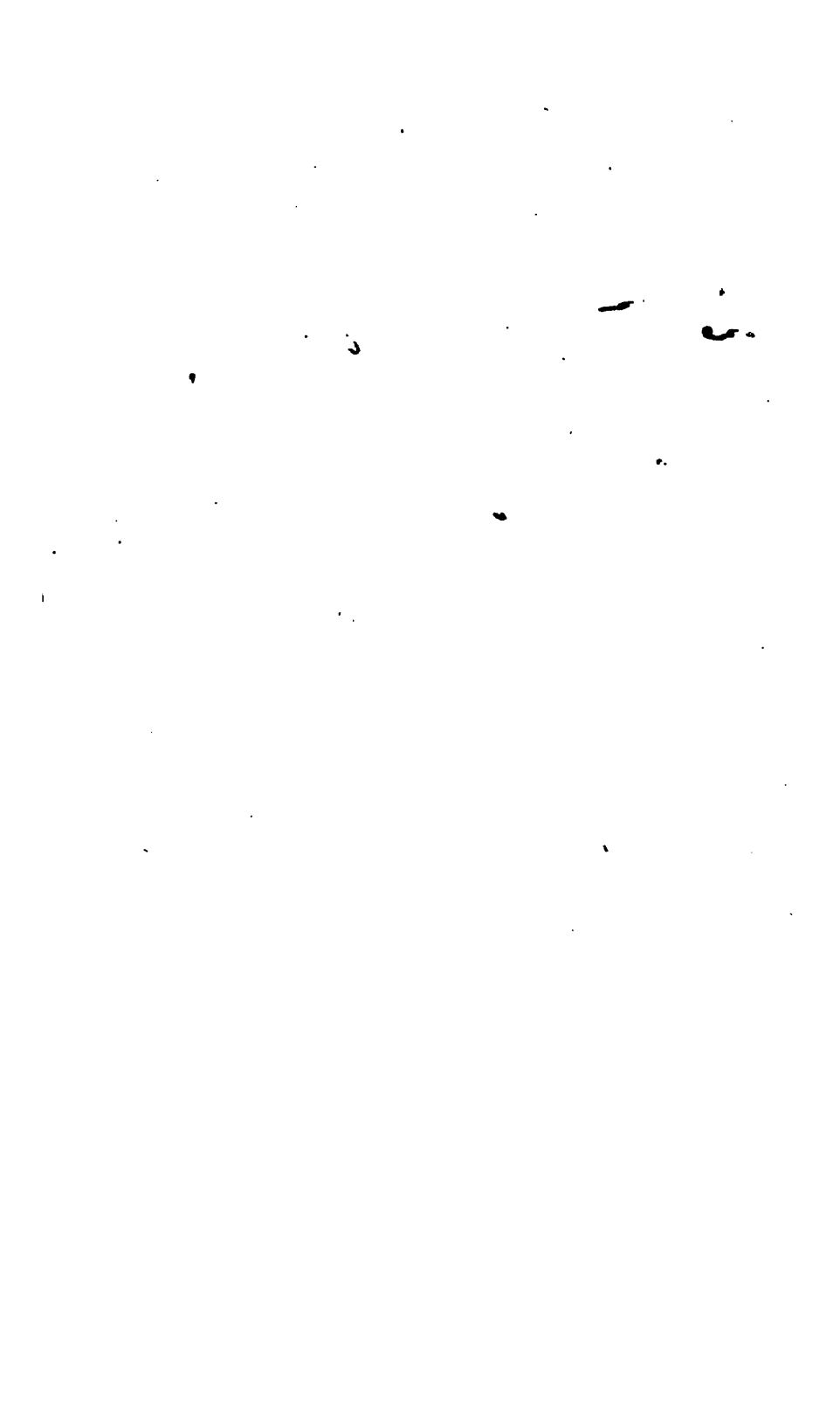
$$1 \quad \frac{a+b}{2} + x = \sqrt{ab}$$

$$\frac{(a+b)^2}{4} + (a+b)x + x^2 = ab$$

$$x^2 + (a+b)x + \frac{(a-b)^2}{4} = 0$$

~~1~~ $\frac{(a+b)^2}{4} > \frac{(a-b)^2}{4}$

$\therefore$ LHS term is $\geq$ RHS term $\therefore$ x is real
two values of x are possible
-ve $\therefore \frac{a+b}{2} > \sqrt{ab}$.



Three prizes, amounting altogether to £100, are distributed, in geometric progression. The largest is £60 more than the least. Find the value of each prize.

Let x, xy, xy^2 be their respective values.

$$\therefore xy^2 = x + 60 \quad \therefore x = \frac{60}{y^2 - 1}$$

$$\text{Also } x + xy + xy^2 = 100$$

$$\text{or } x + xy + x + 60 = 100$$

$$\therefore xy + 2x = 40 \quad \therefore x = \frac{40}{y + 2}$$

$$\therefore \frac{60}{y^2 - 1} = \frac{40}{y + 2}$$

$$\therefore \frac{3}{y^2 - 1} = \frac{2}{y + 2}$$

$$\therefore 3y + 6 = 2y^2 - 2$$

$$\therefore y^2 - \frac{3y}{2} = 4$$

$$y^2 - \frac{3y}{2} + \frac{9}{16} = 4 + \frac{9}{16} = \frac{73}{16}$$

$$y - \frac{3}{4} = \frac{\pm \sqrt{73}}{4}$$

$$y = \frac{3 \pm \sqrt{73}}{4}$$

There are three numbers in geometrical progression, whose sum is 52, and the sum of the extremes is to the mean as 10 : 3. Find the numbers.

Let x, xy, xy^2 , be the numbers,

$$\text{Then } x + xy + xy^2 = 52$$

$$\therefore x + xy^2 = 52 - xy$$

$$x + xy^2 : xy :: 10 : 3$$

$$\text{or } 52 - xy : xy :: 10 : 3$$

$$\therefore 52 : xy :: 13 : 3 \quad \therefore xy = 12$$

$$\text{Also } 3x + 3xy^2 = 10xy$$

$$\therefore 3 + 3y^2 = 10y, \text{ \&c.}$$

$$y = 3, x = 4$$

$$\text{The numbers are 4, 12, 36.}$$

$$\text{or } x = \frac{1}{3}, y = 36$$

$$\text{or } 36, 12, 4.$$

There are three numbers in geometrical progression, whose sum is 14; and the sum of the first and second is to the sum of the second and third as 1 : 2. Find the numbers.

Let x, xy, xy^2 be the numbers,

$$x + xy : xy + xy^2 :: 1 : 2$$

$$\therefore 1 + y : y + y^2 :: 1 : 2$$

$$\therefore 1 + y : \overline{1 + y} \cdot y :: 1 : 2$$

$$\therefore 1 : y :: 1 : 2$$

$$\therefore y = 2$$

$$\text{and } x(1 + 2 + 4) = 14 \therefore x = 2$$

The numbers are 2, 4, 8.

Find the sum of (n) terms of the series $c^2 - x^2, c + x, \frac{c+x}{c-x}, \&c.$

$$r = \frac{c + x}{c^2 - x^2} = \frac{1}{c - x}$$

$$a = c^2 - x^2$$

$$\therefore S = \frac{ar^n - a}{r - 1} = (c^2 - x^2) \times \frac{\left(\frac{1}{c-x}\right)^n - 1}{\frac{1}{c-x} - 1}$$

$$= \frac{c^2 - x^2}{(c - x)^{n-1}} \times \frac{1 - \frac{c-x}{c-x}}{1 - \frac{c-x}{c-x}}$$

$$= \frac{c + x}{(c - x)^{n-2}} \times \frac{(c-x)^n - 1}{c - x - 1}$$

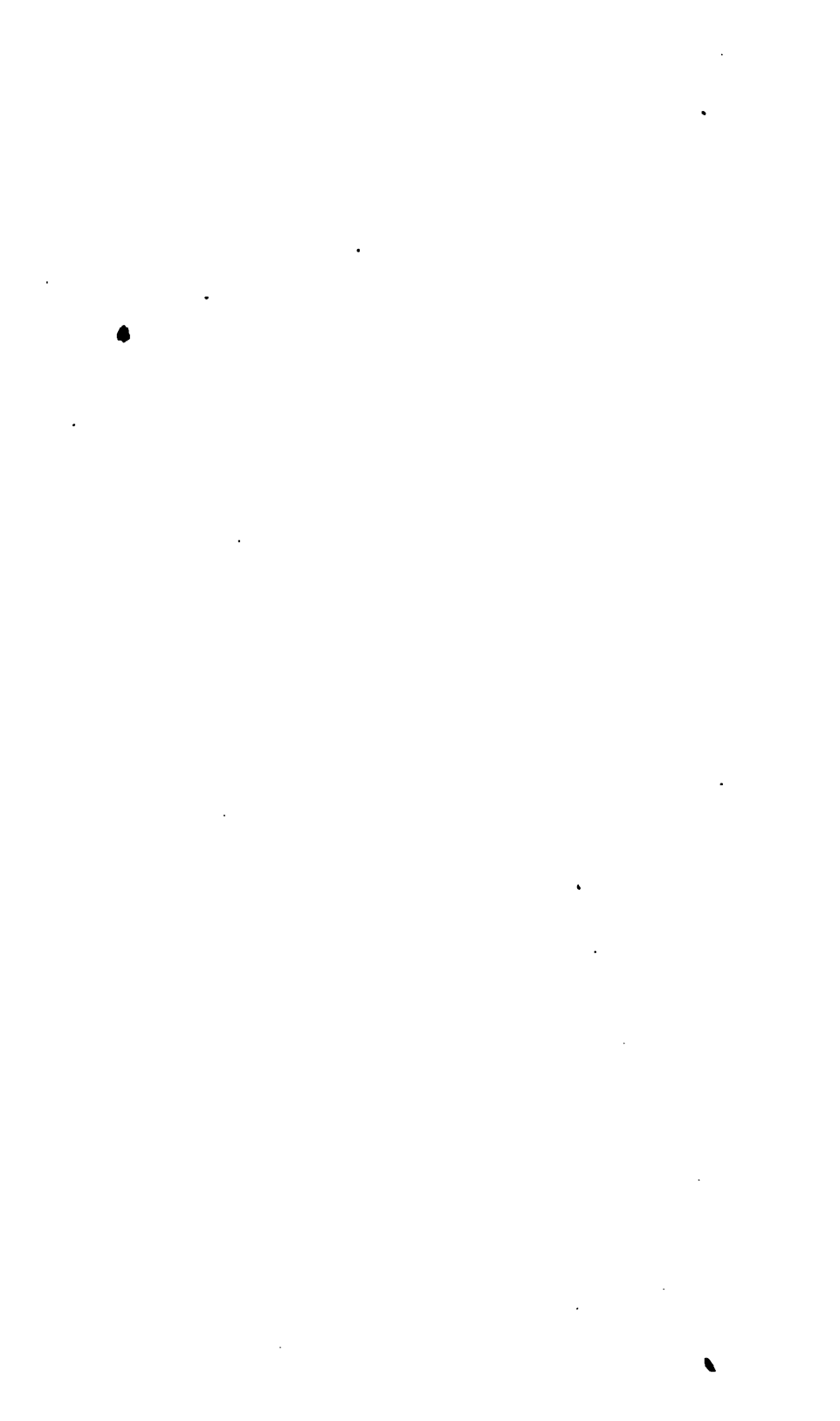
Sum the series, $a, b, \frac{b^2}{a}, \&c.$ both when (n) is finite and infinite.

$$r = \frac{b}{a}$$

$$\text{In the first case, } S = \frac{ar^n - a}{r - 1} = \frac{\frac{b^n}{a^{n-1}} - a}{\frac{b}{a} - 1}$$

$$= \frac{b^n - a^n}{ba^{n-2} - a^{n-1}} = \frac{a^n - b^n}{(a - b) \cdot a^{n-2}}$$





In the second case
$$S = \frac{a}{1-r} = \frac{a}{1-\frac{b}{a}} = \frac{a^2}{a-b}$$

Find the sum of the infinite series, of which the first term is 1, and any one term is equal to the sum of all the following ones. Let the series be $a, ar, ar^2, \dots$ &c.

$$S = \frac{a}{1-r} = \frac{1}{1-r}$$

Then $1 = r + r^2 + r^3 + \dots$ &c.

$$\text{But } 1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

$$\therefore r + r^2 + r^3 + \dots = \frac{r}{1-r} = 1$$

$$\therefore r = 1 - r, \therefore r = \frac{1}{2}$$

And the series is $1 + \frac{1}{2} + \frac{1}{4} + \dots$ &c. in infinitum ($= 2$).

HARMONICAL PROGRESSION.

If the quantities a, b, c be in harmonical progression, their reciprocals are in arithmetical progression: for

$$\text{Let } a-b : b-c :: a : c$$

$$\text{Then } ac - bc = ab - ac$$

$$\therefore \frac{ac}{abc} - \frac{bc}{abc} = \frac{ab}{abc} - \frac{ac}{abc}$$

$$\text{or } \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in arithmetical progression.}$$

Thus 3, 4, 6 are in harmonical progression, because $1-4 : 4-3 :: 6 : 3$; and $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ are in arithmetical progression; for $\frac{1}{3} - \frac{1}{4} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$.

Also, 2, 3, 4, 5 are in arithmetical progression; and $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ are in harmonical progression; for they are to each other as the numbers 60, 40, 30, and 24. If four musical strings of equal tension and thickness be made to sound together, the most perfect harmony they can produce will be when their lengths are in the proportion of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, or as 60, 40, 30, and 24. The same reasoning applies to any number of strings. Whence this progression receives its name.

What is the harmonic mean between a and b ?

Let x be the mean,

$$\therefore a-x : x-b :: a : b$$

$$\therefore ba-bx=ax-ab \quad \therefore 2ab=a+b \cdot x$$

$$\therefore x=\frac{2ab}{a+b}$$

Now we find the arithmetic mean to be $\frac{a+b}{2}$; and the geometric mean is $\sqrt{ab}$; and it is manifest that the three means, $\frac{a+b}{2}$, $\sqrt{ab}$, and $\frac{2ab}{a+b}$ are in geometrical progression; for the product of the two extremes $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$, is equal to the square of the middle term, $\sqrt{ab}$.

If it be required to find the n^{th} term of an harmonical series, a, b, c : Invert the terms, and we then have $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ an arithmetical series; of which the first term is $\frac{1}{a}$, the common difference is $\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$; $\therefore$ the n^{th} term is $\frac{1}{a} + n-1 \times \frac{a-b}{ab}$, the inverse of which expression is the n^{th} term of the harmonic series.

Solve the equation $x^3+11x^2+36x+36=0$; where the values of x are in harmonical progression.

Suppose a, b, c to be the values of x , then $a-b : b-c :: a : c$ $\therefore ac-bc=ab-ac$ $\therefore 3ac=ab+bc+ac$, but $ab+bc+ac=36$, the third term in the equation.

$\therefore 3ac=36$ $\therefore ac=12$; also $-abc=36$, the last term in the equation $\therefore b=-3$.

Divide the proposed equation by $x+3$, and we then have to solve the quadratic $x^2+8x+12=0$, the roots of which equation are -2 and -6 ; so that the roots of the original equation are $-2, -3, -6$.





LOGARITHMS

To expand a number, a^x , into a series involving the logarithm (x) and base (a).

Let $a = 1 + b$; then $a^x = \overline{1+b}^x$.

$$\begin{aligned} &= 1 + xb + x \cdot \frac{x-1}{2} b^2 + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} b^3 + x \cdot \frac{x-1}{2} \\ &\cdot \frac{x-2}{3} \cdot \frac{x-3}{4} \cdot b^4 + \&c. = 1 + x \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \&c. \right) \\ &+ \frac{x^2}{1 \cdot 2} (b^2 - b^3 + \&c.) + Rx^3 + \&c. \end{aligned}$$

Let $b - \frac{b^2}{2} + \frac{b^3}{3} - \&c. = M$, and $\frac{1}{2} (b^2 - b^3 + \&c.) = N$,

$\therefore \overline{1+b}^x = 1 + Mx + Nx^2 + Rx^3 + \&c.$ Then will

$$\overline{1+b}^{2x} = 1 + M \times 2x + N \times (2x)^2 + R \cdot (2x)^3 + \&c.$$

$$\text{But } \overline{1+b}^{2x} = \overline{1+b}^x \times \overline{1+b}^x;$$

$$1 + Mx + Nx^2 + Rx^3 + \&c. = \overline{1+b}^x$$

$$1 + Mx + Nx^2 + Rx^3 + \&c. = \overline{1+b}^x$$

$$\left. \begin{aligned} &1 + Mx + Nx^2 + Rx^3 + \&c. \\ &+ Mx + M^2x^2 + MNx^3 + \&c. \\ &+ Nx^2 + MNx^3 + \&c. \\ &+ Rx^3 + \&c. \end{aligned} \right\} \overline{1+b}^x \times \overline{1+b}^x$$

$\therefore 1 + 2Mx + x^2(M^2 + 2N) + x^3(2R + 2MN) + \&c.$
 $= 1 + 2Mx + N \cdot (2x)^2 + R \cdot (2x)^3 + \&c.$ Then equating the corresponding terms, we have $4Nx^2 = (M^2 + 2N)x^2$.

$$\therefore 2N = M^2 \quad \therefore N = \frac{M^2}{1 \cdot 2};$$

$$\text{Also } 8Rx^3 = (2R + 2MN)x^3$$

$$\therefore 6R = 2MN \quad \therefore R = \frac{MN}{3} = \frac{M^3}{1 \cdot 2 \cdot 3}$$

In like manner $S = \frac{M^4}{1 \cdot 2 \cdot 3 \cdot 4}$, &c. Hence $\overline{1+b^x}$, or u
 $1 + Mx + \frac{M^2 x^2}{1 \cdot 2} + \frac{M^3 x^3}{1 \cdot 2 \cdot 3} +$, &c. where $M = \overline{a-1} - \frac{1}{2} \overline{a-1}$
 $+ \frac{1}{3} \overline{a-1}^2 - \frac{1}{4} \overline{a-1}^3 +$ &c.

Having given a number; to find its logarithm.

Let the given number be a^x , x being its logarithm in system of which (a) is the base.

Assume $a = 1 + m$; and let $a^x = 1 + n$

Then $\overline{1+m}^x = 1 + n$; $\therefore \overline{1+m}^{xy} = \overline{1+n}^y$.

Now $\overline{1+m}^{xy} = 1 + xy \cdot m + xy \cdot \frac{xy-1}{2} m^2 + xy \cdot \frac{xy-2}{3} m^3 + xy \cdot \frac{xy-1}{2} \cdot \frac{xy-2}{3} \cdot \frac{xy-3}{4} m^4 +$ &c.

And $\overline{1+n}^y = 1 + yn + y \cdot \frac{y-1}{2} n^2 + y \cdot \frac{y-1}{2} \cdot \frac{y-2}{3} n^3 +$
 $\frac{y-2}{3} n^3 +$, &c.

Then equating the corresponding terms, and dividing both sides of the equation by (y), we have

$$\begin{aligned} x(m + \frac{xy-1}{2} \cdot m^2 + \frac{xy-1}{2} \cdot \frac{xy-2}{3} m^3 + \frac{xy-1}{2} \cdot \frac{xy-2}{3} \cdot \frac{xy-3}{4} m^4 + \&c) = n + \frac{y-1}{2} \cdot n^2 + \frac{y-1}{2} \cdot \frac{y-2}{3} \cdot n^3 + \frac{y-1}{2} \cdot \frac{y-2}{3} \cdot \frac{y-3}{4} \cdot n^4 + \&c. \end{aligned}$$

Now as this equation is true for all possible values; suppose $y = 0$; then we have

$$x(m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} +, \&c.) = n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} +$$

$$\therefore x = \frac{n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} +, \&c.}{m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} +, \&c.}$$

$$n - \frac{n^2}{2} + \frac{n^3}{3} -, \&c.$$

a is assumed $= 1 + m$, for if it were not greater than unity it could not be used as the base of a logarithmic series, since all its powers would $= 1$, & $\therefore a^x$ would not represent any number and a^x is assumed $= 1 + n$, for if it were not $=$ ^{or less} than unity, x would $= 0$. see part I p. 7 at bottom of the page.

c. Let m be a proper fraction in order
that it may be $< \text{unity}$.

Thus we have found an expression for (x) the logarithm, in terms of the number $(n+1)$, and base (a) . But in order to reduce this expression to calculation we must transform it into a series, of which the succeeding terms more rapidly decrease.

$$\text{Let } (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \dots = M;$$

$$\text{Then since log of } (1+n) = \frac{1}{M} \left(n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} + \&c. \right)$$

$$\therefore \text{log. of } (1-n) = \frac{1}{M} \left(-n - \frac{n^2}{2} - \frac{n^3}{3} - \frac{n^4}{4} - \&c. \right)$$

$$\therefore \text{log. } \overline{1+n} - \text{log. } \overline{1-n} = \frac{1}{M} \left(2n + \frac{2n^3}{3} + \frac{2n^5}{5} + \&c. \right)$$

$$\text{or log. } \frac{1+n}{1-n} = \frac{2}{M} \left(n + \frac{n^3}{3} + \frac{n^5}{5} + \&c. \right)$$

Now to transform this into a converging series, let
 $n = \frac{1}{P+1},$

$$\therefore 1+n = 1 + \frac{1}{P+1} = \frac{P+2}{P+1};$$

$$\text{and } 1-n = 1 - \frac{1}{P+1} = \frac{P}{P+1}$$

$$\therefore \frac{1+n}{1-n} = \frac{P+2}{P+1} \times \frac{P+1}{P} = \frac{P+2}{P}; \quad \therefore \text{log. } \frac{1+n}{1-n} =$$

$$\text{Log. } \frac{P+2}{P} = \frac{2}{M} \left(\frac{1}{P+1} + \frac{1}{3 \cdot P+1} + \frac{1}{5 \cdot P+1} + \&c. \right)$$

$$\therefore \text{log. } (P+2) = \frac{2}{M} \left(\frac{1}{P+1} + \frac{1}{3 \cdot P+1} + \frac{1}{5 \cdot P+1} + \&c. \right)$$

+ log. P . $\frac{1}{M}$ is called the modulus of the system; and

its value evidently depends upon the base (a) .

If $\frac{1}{M}$ and $\frac{1}{M'}$ be moduli of two different systems, the log. of a *given number* in the first system is to its logarithm in the second $\therefore \frac{1}{M} : \frac{1}{M'}$.

In the Naperean (or hyperbolic) system, the modulus is unity. Hence in any other system, where the modulus is $\frac{1}{M}$, the log. of a *given number* = hyperb. log. multiplied into the modulus.

To find the hyperbolic logarithm of 3.

Here $P = 1$; $M = 1$,

$$\begin{aligned}\text{hyp. log. (3)} &= 2 \left(\frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \&c. \right) + \log. 1 (= 0) \\ &= 1 + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 16} + \frac{1}{7 \cdot 64} + \&c. \\ &= 1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448} + \&c. = 1.096 \&c.\end{aligned}$$

To find hyp. log. of 4.

$P = 2$, $M = 1$.

$$\text{hyp. log. (4)} = 2 \left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \&c. \right) + \log. (2)$$

But $\log. 4 = \log. 2^2 = 2 \log. 2$.

$$\begin{aligned}\therefore 2 \log. 2 - \log. 2 (= \log. 2) &= 2 \left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \&c. \right) \\ &= .6931472\end{aligned}$$

$$\therefore \text{hyp. log. 4} = 2 \log. 2 = 1.3862944.$$

By application of the above expression for the log. $(P+2)$ we can find the hyp. log. of 5, 7, 11, 13, 17, 19, &c. &c.

The log. of 6 = $\log. (2 \times 3) = \log. 2 + \log. 3$

The log. of 8 = $\log. (2^3) = 3 \log. 2$

The log. of 9 = $\log. (3^2) = 2 \log. 3$

The hyp. log. of 10 = $\log. 5 + \log. 2 = 2.3025851$

Having found the hyp. log. of 10, we can find the value of the modulus of the common system. For since the com-

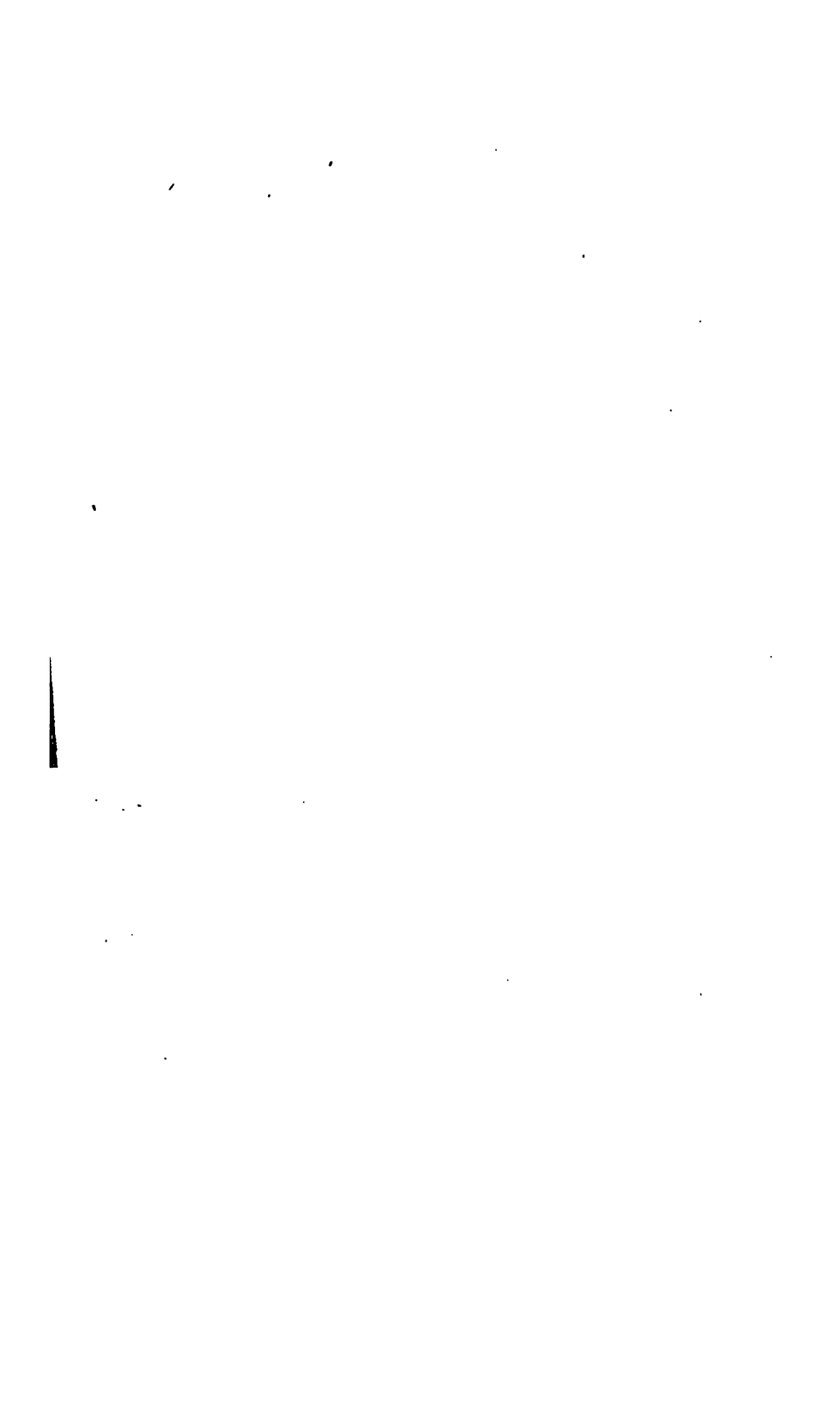
mon log. of 10 = $\frac{1}{M} \times \text{hyp. log. of 10}$,

$$\therefore 1 = \frac{1}{M} \times 2.3025851 \quad \therefore \frac{1}{M} = .43429448$$

And the log. of any number $(P+2)$ in the common system, =

$$\begin{aligned}.86858896 \times \left(\frac{1}{P+1} + \frac{1}{3 \cdot (P+1)^3} + \frac{1}{5 \cdot (P+1)^5} + \&c. \right) \\ + \log. (P).\end{aligned}$$





It was shewn that $a^x = 1 + Mx + \frac{M^2 x^2}{1 \cdot 2} + \frac{M^3 x^3}{1 \cdot 2 \cdot 3} + \&c.$

$$\text{Let } x = \frac{1}{M}$$

$$\text{Then } a^{\frac{1}{M}} = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \&c. = 2.71828182$$

Hence, in *any system*, the logarithm of the given number 2.7182818 is the modulus of that system. Therefore, in Napier's system, where the modulus is unity, 2.7182818 must be the base. This also appears from making $M=1$, and $x=1$, in the equation $a^x = 1 + Mx + \frac{M^2 x^2}{1 \cdot 2} + \&c.$ Call

the base of this system (e); then $e = a^{\frac{1}{M}}$, and $e^M = a$.

Since $a^{\frac{1}{M}}$ is a given quantity, where $M = (a-1) - \frac{1}{2}(a-1)^2 + \&c.$

If we take $a=e$, $M=1$; and consequently $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \&c. \text{ in infinitum} = 1.$

INTEREST.

WHEN interest, left unpaid, is added to the principal, the whole sum thus increased bears what is called *Compound Interest*. Thus if (r) be the interest of £1 for 1 year, $r+1$ = amount at the end of 1 year.

£ £

$\therefore 1 : r+1 :: r+1 : \overline{r+1}^2$ = amount of £1 in two years

In like manner $\overline{r+1}^3$ = amount of £1 in three years.

And $\overline{r+1}^n$ = amount of £1 in (n) years.

$\therefore P \times \overline{r+1}^n$ = (if $r+1=R$) PR^n the amount of P in (n) years.

From this equation $PR^n = M$ (the amount); any two terms may be given to find the third.

Hence the present value (P) of any sum (M) due (n) years

hence is $= \frac{M}{R^n}.$

Find the amount of £300 in 4 years at 4 per cent. annum compound interest.

$$P=300, r=.04 \quad \therefore R=1.04$$

$$n=4$$

$$M=300 \times 1.04^4 \quad \therefore \log. M = \log. 300 + 4 \log. 1.04$$

$$= 2.5452545$$

$$\therefore M=350.95 = £350 : 19s. : 6d.$$

What is the amount of £15 : 10s. for 9 years at $3\frac{1}{2}$ per cent. compound interest?

$$P=15.5, n=9 \quad r=.035$$

$$M=15.5 \times 1.035^9 \quad R=1.035$$

$$\therefore \log. M = \log. 15.5 + 9 \log. 1.035$$

$$= 1.1903317 + .1344627$$

$$= 1.3247944$$

$$\therefore M=21.125 = £21 : 2s. : 6d.$$

What sum, placed out at compound interest, will amount to £1000 in 20 years at 5 per cent?

$$M=PR^n; 1000 = P \times 1.05^{20} \quad \therefore P = \frac{1000}{1.05^{20}}$$

$$\therefore \log. P = \log. 1000 - 20 \log. 1.05$$

$$= 2.576214$$

$$\therefore P=376.89 = £376 : 17s. : 9\frac{1}{2}d.$$

At what rate of interest must £300 be placed out to amount to £350 : 19s. : 2d. in 4 years.

$$M=PR^n;$$

$$\begin{array}{ccccccc} s. & d. & s. & £. & £. & s. & d. & £. \\ 19 & : & 2 & \text{is } 19.1666 = .95833; & \therefore & 350 : 19 : 2 = 350.95 \end{array}$$

$$350.95833 = 300 \times R^4$$

$$\therefore \log. 350.95833 = \log. 300 + 4 \log. R$$

$$\therefore \log. R = \frac{350.95833 - \log. 300}{4}$$

$$= .017015525 \quad \therefore R=1.04$$

$$\therefore r=.04$$

and 1 : 100 :: .04 : 4, the rate per cent.





In how many years will £500 amount to £800 at 5 per cent?

$$M = PR^n$$

$$800 = 500 \times 1.05^n$$

$$\therefore n \times \log. 1.05 = \log. 800 - \log. 500 = .20412$$

$$\text{or } n \times .0211893 = .20412$$

$$\therefore n = \frac{.20412}{.0211893} = 9.63269.$$

In what time will a sum of money double itself at 5 per cent. compound interest?

$$M = PR^n; \text{ here } M = 2P$$

$$\therefore 2P = PR^n \quad \therefore 2 = R^n \quad \therefore n \log. R = \log. 2.$$

$$\therefore n = \frac{\log. 2}{\log. R} = \frac{\log. 2}{\log. 1.05}$$

$$\frac{.3010300}{.0211893} = 14.2067 \text{ years.}$$

In what time will a sum double itself at $3\frac{1}{2}$ per cent.?

$$M = PR^n; \text{ here } M = 2P \quad \therefore 2P = PR^n$$

$$R = 1.035 \quad \therefore R^n = 2$$

$$\therefore n \times \log. R = \log. 2$$

$$\therefore n = \frac{\log. 2}{\log. 1.035} = \frac{.3010300}{.0149403} = 20.149$$

If the interest be payable half yearly, we must take $r' = \frac{r}{2}$ the interest of £1 for half a year, and $1+r' (= R)$ amount of £1 in half a year; $n' = 2n$.

What is the amount of £600 in six years at $4\frac{1}{2}$ per cent. Compound interest, interest payable half yearly?

$$r = .045$$

$$r' = \frac{r}{2} = .0225 \quad \therefore R = 1.0225$$

$$M (= PR^n) = 600 \times 1.0225^{12}$$

$$\text{Log. } M = \log. 600 + 12 \log. 1.0225 = 2.8941109$$

$$\therefore M = \begin{array}{ccc} \text{£} & & \text{£} \\ 783 & \cdot 63 & = 783 : 10 : 7 \end{array}$$

Find the amount of £1200 for 10 years at 6 per cent., interest payable half yearly.

Answer, £2167 : 6s. : 0d.

If (A) be an annuity left unpaid for (n) years, find its amount and present value.

At the end of the first year A becomes due, and the amount of it in $(n-1)$ years is AR^{n-1}

At the end of 2d year A becomes due, and its amount at the end of $(n-2)$ years is AR^{n-2}

From the end of the 3d year A accumulates for $(n-3)$ years, and its amount is AR^{n-3}

From the end of $n-1$ years, the annuity (A) bears interest for 1 year, and its amount is AR

At the end of the n^{th} year there is due..... A

$\therefore$ sums due at expiration of (n) years is $A + AR + AR^2$

+ &c. ... AR^{n-1} ; the amount of these sums is $\frac{AR^n - A}{R - 1}$;

and the present value of this sum is $\frac{AR^n - A}{R^n \times (R - 1)}$

What is the amount of an annuity of £50 for 7 years at 4 per cent.?

$$M = \frac{AR^n - A}{R - 1} = \frac{50 \times (1.04)^7 - 1}{.04}$$

Now $\log. (1.04)^7 = 7 \times \log. 1.04 = .11921 = \log. 1.3160$

$$\therefore M = \frac{50 \times .3160}{.04} = £395.$$

What is the amount of an annuity of £40 for 5 years at 5 per cent.?

$$M = \frac{AR^n - A}{R - 1} = \frac{40 (1.05)^5 - 1}{.05}$$

Now $\log. 1.05^5 = 5 \log. 1.05 = 1.059465 = \log. 1.2763$

$$\therefore M = \frac{40 \times .2763}{.05} = £221 : 0s. : 9\frac{1}{2}d.$$

For at compound interest $PR^n = M$

$$\therefore P = \frac{M}{R^n} = \frac{AR^n - A}{R^n(R-1)}$$



What is the amount of an annuity of £50 in 20 years at $3\frac{1}{4}$ per cent.?

$$M = \frac{AR^n - A}{R - 1} = \frac{50 \times (1.035)^{20} - 1}{.035}$$

$$\text{Log. } (1.035)^{20} = 20 \log. 1.035 = .298806 = \log. 1.989784$$

$\therefore (1.035)^{20}$ is the same as 1.9898

$$\therefore M = \frac{50 \times .989784}{.035} = \text{£}1414 : 0s. : 0d.$$

Find the present value of an annuity of £50 for 20 years at $3\frac{1}{4}$ per cent.

$$\text{Present value} = \frac{A(R^n - 1)}{R^n \times (R - 1)} = \frac{50(1.035)^{20} - 1}{1.035^{20} \times .035}$$

$$\text{The amount is } \frac{50 \times .989784}{.035}$$

$$\therefore \text{Present value } (P) = \frac{1413.98}{1.989784}$$

$$= 710.62 = \text{£}710 : 12s. : 4d. \frac{3}{4}$$

If in the expression $\frac{AR^n - A}{R^n(R - 1)} \left(= \frac{A - \frac{A}{R^n}}{R - 1} \right)$, (n) be infinite, then $\frac{A}{R^n} = 0$; and the present value of a freehold estate, whose yearly rent is A , $= \frac{A}{R - 1} = \frac{A}{r}$.

What is the price of a freehold estate whose yearly rent is a £, reckoning 4 per cent.?

$$\frac{A}{r} = \frac{a}{.04} = \frac{100a}{4} = 25a = 25 \text{ years' purchase.}$$

How much stock in 3 per cent consols when they are at $70\frac{3}{8}$ ought to be given for an estate of £240 a-year?

$\dagger \quad 70\frac{3}{8} : 1 :: 3 : r = \frac{3 \times 8}{563} = \frac{24}{563} = \cdot 04263$, then $\frac{A}{r}$
 $= \frac{240}{\cdot 04263} = £5629 : 16s. : 9d.$ By Proportion we find
 the stock which this sum will purchase when consols
 are at $70\frac{3}{8}$;

Thus $70\cdot375 : 5629\cdot8375 :: 100 : \frac{562983\cdot75}{70\cdot375}$, the num-
 ber of hundred pounds stock which £5629 16s. 9d. will
 purchase in the 3 per cents. when they are at $70\frac{3}{8}$.

EXAMPLES.

1. What is the amount of £575 in 7 years at 4 per cent.
 compound interest?

Answer, £756 13s. 2½d.

2. What sum placed out to compound interest will
 amount to £1000 in 15 years, at 5 per cent.?

Answer, £481 0s. 4¾d.

3. What is the rate of interest when an annuity of £120,
 forborne 8 years, amounts to 1200?

Answer, 6·287.

4. What annuity, to continue 20 years, may be pur-
 chased for £1000, at $3\frac{1}{2}$ per cent.?

Answer, £70 7s. 2d.

5. For what number of years can an annuity of £100 be
 purchased with £600, supposing the rate of interest 5 per
 cent.?

Answer, 7.

6. At what rate of interest can an annuity of £50 be
 purchased with £400, to continue 10 years?

Answer, 4·2775 per cent.

* Now this is to find the interest for one pound.

$$\begin{aligned}
 \times \quad S &= \overline{2a+n-1} \cdot \frac{n}{2} = \overline{2+n-1} \cdot \frac{n}{2} \\
 &= \frac{n \cdot n+1}{2}
 \end{aligned}$$

ON NOTATION, NUMBERS, &c.

FIGURATE NUMBERS.

IF several orders of numbers be formed from each other, in such a manner that the sum of the first (n) terms of one order shall be equal to the n^{th} term of the next; the numbers thus arising are called *Figurate numbers*. Thus if we begin with the least possible number (1) and call

1, 1, 1, 1, 1, 1 the 1st order.

1 2 3 4 5 (n), is the 2d order.

1 3 6 10 15 . . . $n \cdot \frac{n+1}{2}$, the 3d order.

1 4 10 20 35 $n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}$, the 4th order.

1 5 15 35 70 $n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}$, the 5th

&c. &c. &c. &c.

order.

The n^{th} term in any order is found by induction;

Thus the 4th term in the 3d order is $\frac{4 \times 5}{1 \times 2} = 10$,

And the 5th term in the 3d order is $\frac{5 \times 6}{1 \times 2} = 15$.

$\therefore$ the n^{th} term in the 3d order is $n \cdot \frac{n+1}{2}$ *

Also the 5th term in the 4th order is $\frac{5 \times 6 \times 7}{1 \times 2 \times 3} = 35$,

* This expression also arises from taking the sum of (n) terms of the preceding arithmetical series.

$\therefore$ the n^{th} term in the 4th order is $\frac{n \cdot \overline{n+1} \cdot \overline{n+2}}{1 \cdot 2 \cdot 3}$

In like manner the n^{th} term in the 5th order is found to be $n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}$.

And the n^{th} term in the p^{th} order will be $n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} \dots \frac{n+p-2}{p-1}$.

The $(n+1)^{\text{th}}$ term of the p^{th} order is to the n^{th} term of the $(p+1)^{\text{th}}$ order, as (p) to (n) .

For the $(n+1)^{\text{th}}$ term of the p^{th} order is $\frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \dots \overline{n+p-1}}{1 \cdot 2 \cdot 3 \cdot 4 \dots p-1}$; and the n^{th} term of

the $(p+1)^{\text{th}}$ order is $\frac{n \cdot \overline{n+1} \cdot \overline{n+2} \dots \overline{n+p-1}}{1 \cdot 2 \cdot 3 \dots p-1 \cdot p}$, and

these two expressions are to each other as $1 : \frac{n}{p} :: p : n$.

Let $p=3$, and $n=5$ $\therefore n+1=6$; and the 6th term of the 3d order = 21; also the 5th term of the 4th order = 35; and $21 : 35 :: 3 : 5$ ($:: p : n$).

If the $(n+1)^{\text{th}}$ term of the p^{th} order be added to the n^{th} term of the $(p+1)^{\text{th}}$ order, their sum is the $(n+1)^{\text{th}}$ term of the $(p+1)^{\text{th}}$ order.

For the $(n+1)^{\text{th}}$ term of the p^{th} order = $\frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \dots \overline{n+p-1}}{1 \cdot 2 \cdot 3 \dots p-1} \times \frac{p}{p}$; and this quantity

added to the n^{th} term of the $(p+1)^{\text{th}}$ order is manifestly equal to $\frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \dots \overline{n+p-1} \cdot \overline{n+p}}{1 \cdot 2 \cdot 3 \dots p-1 \cdot p}$

which is the $(n+1)^{\text{th}}$ term of the $(p+1)^{\text{th}}$ order.



POLYGONAL NUMBERS.

If a series of numbers be so formed that the n^{th} term shall express the sum of an arithmetical series whose first term = 1 the numbers are called *polygonal* numbers.

1. Let the common difference of the arithmetical series = o ; then the sum = n : and the series of polygonal number is

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \dots 1 \quad \text{1st order.}$$

2. Let the common difference = 1; then the sum = $2 + n - 1 \cdot 1 \times \frac{n}{2} = n \cdot \frac{n+1}{2}$; and by substituting successively 1, 2, 3, 4, &c. for (n) , the series becomes

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21 \dots n \cdot \frac{n+1}{2}; \quad \text{2d order.}$$

3. Let common difference of the arithmetical series = 2, then the sum = $2 + n - 1 \cdot 2 \times \frac{n}{2} = n^2$; and by substituting as before for (n) the series becomes

$$1, \quad 4, \quad 9, \quad 16, \quad 25 \dots n^2; \quad \text{3d order.}$$

In like manner if the common difference of the arithmetical series be 3, we have

$$1, \quad 5, \quad 12, \quad 22, \quad 35, \dots \frac{3n^2 - n}{2}; \quad \text{4th order.}$$

$$1, \quad 6, \quad 15, \quad 28, \quad 45 \dots \frac{4n^2 - 2n}{2}; \quad \text{5th order.}$$

$$\text{Hence the } n^{\text{th}} \text{ term of the 6th order} = \frac{(6-1)n^2 - (6-3)n}{2}$$

$$\text{And the } n^{\text{th}} \text{ term of the } p^{\text{th}} \text{ order} = \frac{(p-1)n^2 - (p-3)n}{2}$$

It appears that the second order of polygonal numbers is the same as the third order of figurative numbers: and this is called the series of triangular numbers, because each term denotes the number of units that may be placed at equal distances from each other in the form of an equi-

lateral triangle. And the sum of the series $(1 + 3 + \overset{6}{10} + \&c. \dots n \cdot \frac{n+1}{2}) = n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}$; which denotes the number of balls in a triangular pyramid, where one ball is placed at the top, and (n) expresses the number of balls in each side of the base. Thus, if shot be heaped together in the form of this triangular pile, and 40 shot form one side of the base, the number of shot in the pile is $40 \times \frac{40+1}{2} \times \frac{40+2}{3} = 11480$.

The third order of polygonal numbers, 1, 4, 9, 16 $\dots n^2$, is a series of square numbers, and the sum of this series expresses the number of balls that can be formed into a square pyramid; where one ball is placed at the top, and (n) denotes the number of balls in each side of the base.

The sum of the 4th order of polygonal numbers, 1, 5, 12, &c. expresses the number of balls that can be formed into an equilateral pyramid, of which the top consists of one ball, and its base has the form of an equilateral pentagon, made by joining a square to an equilateral triangle.

It appears that if we take a series of the natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, the sum of (n) terms of this series is the n^{th} term of the second order of polygonal numbers; and (n) of the alternate terms, 1, 3, 5, 7, &c. become the n^{th} term of the third order of polygonal numbers. Also, if we take every third term, 1, 4, 7, 10, 13, &c. the sum of (n) terms of this series expresses the n^{th} term of the 4th order. In like manner the fifth order is formed by the addition of every fourth term, 1, 5, 9, 13, &c. beginning with the first; the 6th order by the addition of every fifth term, &c. &c.

ON INDETERMINATE PROBLEMS.

IN the solution of equations it is generally understood that there are as many independent equations as there are unknown quantities. If the number of unknown quantities exceed the number of independent equations the problem is said to be *indeterminate*.

for it = n^{th} term of 4th order of figurate
number = $n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}$ Ref. p. 81.

$$\begin{aligned}
 * x &= \mu + y = q + r + 14r - 5 = q + 15r - 5 = 4r - 1 + \\
 &= 19r - 6.
 \end{aligned}$$

For instance, if it were required to find all the *positive integral* values of (x) and (y) from the equation $5x + 11y$

$$= 254; x = \frac{254 - 11y}{5} = 50\frac{4}{5} - 2\frac{y}{5}, \text{ a whole number, be-}$$

cause x is, by hypothesis, a whole number. Hence $\frac{4}{5} -$

$\frac{y}{5}$ is a whole number; let then $\frac{4}{5} - \frac{y}{5} = p \therefore 4 - y = 5p$

$\therefore y = 4 - 5p$; and $x = 50 - 2y + p = 50 - 8 + 10p + p = 42 + 11p$. Now (p) may be assumed of any value which will allow the values of x and y to be positive and integral. Its value must, in the present case, be either (0) or negative, to make (y) positive.

1. Take $p = 0 \therefore y = 4$; and $x = 42$
2. Take $p = -1 \therefore y = 9$; and $x = 31$
3. Take $p = -2 \therefore y = 14$; and $x = 20$
4. Take $p = -3 \therefore y = 19$; and $x = 9$.

No other values of (p) can be taken if x and y be both Positive.

Given $14x - 19y = 11$; to find the least possible integral values of x and y ,

$$x = \frac{11 + 19y}{14} = y + \frac{5y + 11}{14}, \text{ a whole number; let } \frac{5y + 11}{14} = p$$

$$\therefore 5y + 11 = 14p \therefore y = \frac{14p - 11}{5} = 2p\frac{4p}{5} - 2\frac{1}{5}; \text{ again,}$$

$$\text{let } \frac{4p - 1}{5} = q \therefore 4p = 5q + 1 \therefore p = \frac{5q + 1}{4} = q + \frac{q}{4} + \frac{1}{4};$$

$$\text{let } \frac{q + 1}{4} = r \therefore q = 4r - 1. \text{ Hence } y = 2p - 2 + q = 2q +$$

$$2r - 2 + q = 3q + 2r - 2 = 12r - 3 + 2r - 2 = 14r - 5; \text{ and}$$

$$x = y + \frac{5y + 11}{14} = 14r - 5 + \frac{70r - 25 + 11}{14} = 14r - 5 +$$

$$\frac{70r - 14}{14} = 14r - 5 + 5r - 1 = 19r - 6. \text{ Now let } r = 1$$

$$\therefore x = 19 - 6 = 13; \text{ and } y = 14r - 5 = 9.$$

These are the *least possible* values of x and y ; the number of values of x and y being evidently indefinite. In general the equation $ax+by=c$ admits of an indefinite number of answers. But the number of answers to $ax+by=c$ is limited. The solution in both cases is possible whenever the common divisor of (a) and (b) is a divisor of c . For suppose a and b divisible by (m) ; and let $a=ma$, and $b=mg$,

$\therefore ax+by=c$ becomes

$$mdx+mg y = c$$

$\therefore dx+gy = \frac{c}{m}$; but dx and gy are whole

numbers, $\therefore \frac{c}{m}$ must be a whole number.

If it were proposed to find in how many ways £100 could be paid in guineas and moidores we have

$$21x+27y = 2000$$

$\therefore 7x+9y = \frac{2000}{3}$; hence, from the last

proposition, this problem is impossible.

In how many different ways can I pay £20 in half guineas and half crowns?

Let x = number of half guineas,

y = number of half crowns,

$\therefore 21x+5y = 800$ sixpences, $21x+5y = 800$,

$\therefore x = \frac{800-5y}{21} = 38\frac{2}{21} - \frac{5y}{21}$; let $\frac{2-5y}{21} = p$

$\therefore 2-5y = 21p \therefore y = \frac{2-21p}{5} = \frac{2-p}{5} - 4p$

Let $\frac{2-p}{5} = q \therefore p = 2-5q$

Hence $y = q-4p = q-8+20q = 21q-8$

$x = 38+p = 38+2-5q = 40-5q$



* This question is wrong.

$$\text{for if } 21x + 3y = 2000$$

$$y = 7x + y = \frac{2000}{3} \text{ which is not a whole number}$$

besides none of the answers are right. It should be

"In how many ways can £139.16.0 be paid in guineas & 3 shilling pieces?"

$$\text{then } 21x + 3y = 2796$$

$$x = \frac{2796 - 3y}{21} = 133\frac{1}{7} - \frac{y}{7} \quad \text{etc.}$$

then the same answers come out

$$* \frac{2000}{21} = 133\frac{1}{3}$$

- | | |
|--|--|
| 1. Let $q=1$. . . $\begin{cases} y=13 \\ x=35 \end{cases}$ | 5. Let $q=5$ $\begin{cases} y=97 \\ x=15 \end{cases}$ |
| 2. Let $q=2$. . . $\begin{cases} y=34 \\ x=30 \end{cases}$ | 6. Let $q=6$ $\begin{cases} y=118 \\ x=10 \end{cases}$ |
| 3. Let $q=3$. . . $\begin{cases} y=55 \\ x=25 \end{cases}$ | 7. Let $q=7$ $\begin{cases} y=139 \\ x=5 \end{cases}$ |
| 4. Let $q=4$. . . $\begin{cases} y=\cancel{76} \\ x=20 \end{cases}$ | |

There are then seven different ways in which £20 can be paid in half-guineas and half-crowns.

✕ In how many ways can £140 be paid in guineas and three-shilling pieces?

Let x = the number of guineas,

y = the number of three-shilling pieces,

$$21x + 3y = 2800$$

$$x = \frac{2800 - 3y}{21} = 133 \frac{1}{7} - \frac{y}{7}; \text{ let } \frac{1-y}{7} = p$$

$$\therefore 1-y=7p \quad \therefore 1-7p=y \quad \therefore x=133+p$$

$$\text{Let } p=0 \quad \therefore y=1 \text{ and } x=133$$

$$p=-1 \quad \therefore y=8 \text{ and } x=132$$

$$p=-2 \quad \therefore y=15 \text{ and } x=131; \text{ and by}$$

continuing this mode of substitution, it appears that after taking $p=0$, there are 132 different negative values of p , which will give as many different positive values of x and y ; $\therefore$ there are upon the whole 133 different ways in which £140 may be paid in guineas and three-shilling pieces.

In how many ways can £10 be paid in crowns and seven-shilling pieces?

Let x = number of seven-shilling pieces,

y = number of crowns,

$$\therefore 7x + 5y = 200 \quad \therefore y = \frac{200 - 7x}{5} = 40 - x + \frac{2x}{5}$$

Let $\frac{2x}{5} = p \therefore 2x = 5p \therefore x = 2p + \frac{p}{2}$ if $(\frac{p}{2} = r) = 2p + r = 5r$; and $y = 40 - 5r - 2r = 40 - 7r$

Let $r = 1 \therefore y = 33, x = 5$

$= 2 \therefore y = 26, x = 10$

$= 3 \therefore y = 19, x = 15, \&c.$ It appears

then that r may be any number less than 6; $\therefore$ there are 5 ways in which £10 may be paid in seven shilling pieces and crowns.

A, who is indebted to B 7s. has no money about him but half-guineas, and B has no money but half-crowns. What is the least number of pieces, by exchanging which they can settle the debt?

Suppose A to pay (x) half-guineas,

And to receive from B (y) half-crowns,

$$\therefore 21x - 5y = 14$$

$$\therefore y = \frac{21x - 14}{5} = 4x - 2 + \frac{x - 4}{5}$$

$$\text{Let } \frac{x - 4}{5} = p \therefore x - 4 = 5p \therefore x = 5p + 4$$

$$\text{And } y = 4x - 2 + p = 20p + 16 - 2 + p = 21p + 14$$

Let $p = 0 \therefore \begin{cases} x = 4; \text{ or A must pay 4 half guineas,} \\ y = 14; \text{ and B pays 14 half-crowns.} \end{cases}$

The number of ways in which the debt may be settled is unlimited.

END OF PART II.

APPENDIX.





$$\frac{x}{5} - 14 = 33$$

$$x - 15 = 165$$

$$x = 180$$

APPENDIX.

PROBLEMS IN EQUATIONS.

1. What two numbers are those whose difference is 14, and whose sum is 48 ?

Answer, 31 and 17.

2. A, B, and C contribute £76 ; B contributes £10 more than A ; and C contributes as much as A and B together : how much did each contribute ?

Answer, 14, 24, and 38.

3. A countryman being asked how many eggs he had, said, " If you diminish the number by 15, and then increase it by 21, and then subtract seven from that sum, and next increase it by 19, I shall then have 48 eggs : " how many had he at first ?

Answer, 25.

4. Find the number of guineas A has in his purse, if that number divided by 5, and then increased by 19, amounts to 23 :

Answer, 20.

5. What number of shillings has A, if 13 times that number, diminished by 25 be $= 170$?

Answer, 15.

6. What number is that whose square root diminished by 7 is $= 16$?

Answer, 529.

APPENDIX.

7. A certain number is added to 1290; and the square root of their sum diminished by 29 is $= 71$: find the number.

Answer, 8710.

8. If from A's age you subtract 11, the square root of the remainder increased by 13 is $= 20$: what is A's age?

Answer, 60.

9. What number is that whose square increased by 51 is $= 100$?

Answer, 7.

10. Four times a certain number is divided by 6; and the square root of this last sum diminished by 60 is $= 40$: what is the number?

Answer, 15000.

11. The price of a house is divided by 8, and then multiplied by 50; and the square root of this last sum increased by 10 is $= £60$: find the price of the house.

Answer £400.

12. Find a certain number which is first multiplied by 4; and this amount increased by 56 times the number divided by 7 is equal to the square of the number.

Answer, 12.

13. What two numbers are those whose sum is 262, and whose difference is 144?

Answer, 203 and 59.

14. A and B have each a flock of sheep: if A give B 7 of his B will then have as many as A; but if, instead of this, B gives A 7 of his, A will then have twice as many as B: how many has each?

Answer 35 and 49.

15. Solve the above when the given number is (a) instead of 7.

16. If the apples which A and B have be added together, their sum is 38; but if from the greater number

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twice the less be deducted, the remainder is 5 : how many has each ?

Answer, 27 and 11.

17. If twice the number of pounds received by A be added to the sum received by B, the amount is £48 ; but if the sum received by A be added to 3 times the sum received by B, the amount is £39 : how much did each receive ?

Answer, £6 and £21.

18. If four times the price of 2 horses be divided by 8 the quotient is £20 ; but if three times the price of the best horse be added to five times the cost of the worst, this sum is £158 : find the price of each.

Answer £19 and £21.

19. If half of A's age, increased by 4 times B's age, and the square root of this sum be taken, the number is 10 ; and if 3 times A's age be added to 5 times B's age, this number is 201 : find the age of each.

Answer, 32 and 21.

20. Three times A's money is increased by £100 ; then this sum is divided by B's money, and the square root of this last amount is 10 : but if 600 be deducted from 250 times B's money, and the remainder be divided by A's money, the square root of this last amount is 2 : how much has each person ?

Answer 4 and 100.

21. A farmer, by good management, in the course of a year doubles his stock, with the exception of £100 laid out in expences ; he continues each year to double his stock, with the exception of £100 ; and at the end of 3 years he finds himself 3 times as rich as at first : what was his first stock ?

Answer, £140.

22. A has 6 sons, each of whom is 4 years older than his next younger brother, and the eldest is 3 times as old as the youngest : find their ages.

Answer, 10, 14, 18, 22, 26, 30.

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23. There is a fish whose head measures 9 inches ; the tail is as long as the head and half the back, and the back is as long as the head and tail together : find the length of the back and the tail.

Answer, $\left\{ \begin{array}{l} \text{length of back is 36 inches.} \\ \text{length of tail is 27 inches.} \end{array} \right.$

24. What part of a lease for 99 years is expired if two-thirds of the past years be equal to four-fifths of those to come ?

Answer, 54.

25. Divide the number 50 into two such parts that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other, makes 40.

Answer, 20 and 30.

26. Divide 60 into two such parts that $\frac{1}{4}$ of one part may be $= \frac{1}{8}$ of the other.

Answer, 28 and 32.

27. Divide 100 into two such parts that if $\frac{1}{3}$ of one part be subtracted from $\frac{1}{4}$ of the other the remainder may be 11.

Answer, 24 and 76.

28. A has 72s. and B has 52s. when they sit down to play ; at last A finds he has 3 times as much as B : how many shillings did A win ?

Answer, 21.

29. A gives 4d. a piece to some beggars, and has 16d. remaining ; he would have given them 6d. a piece, but found he had not enough by a shilling : how many beggars are there ?

Answer, 14.

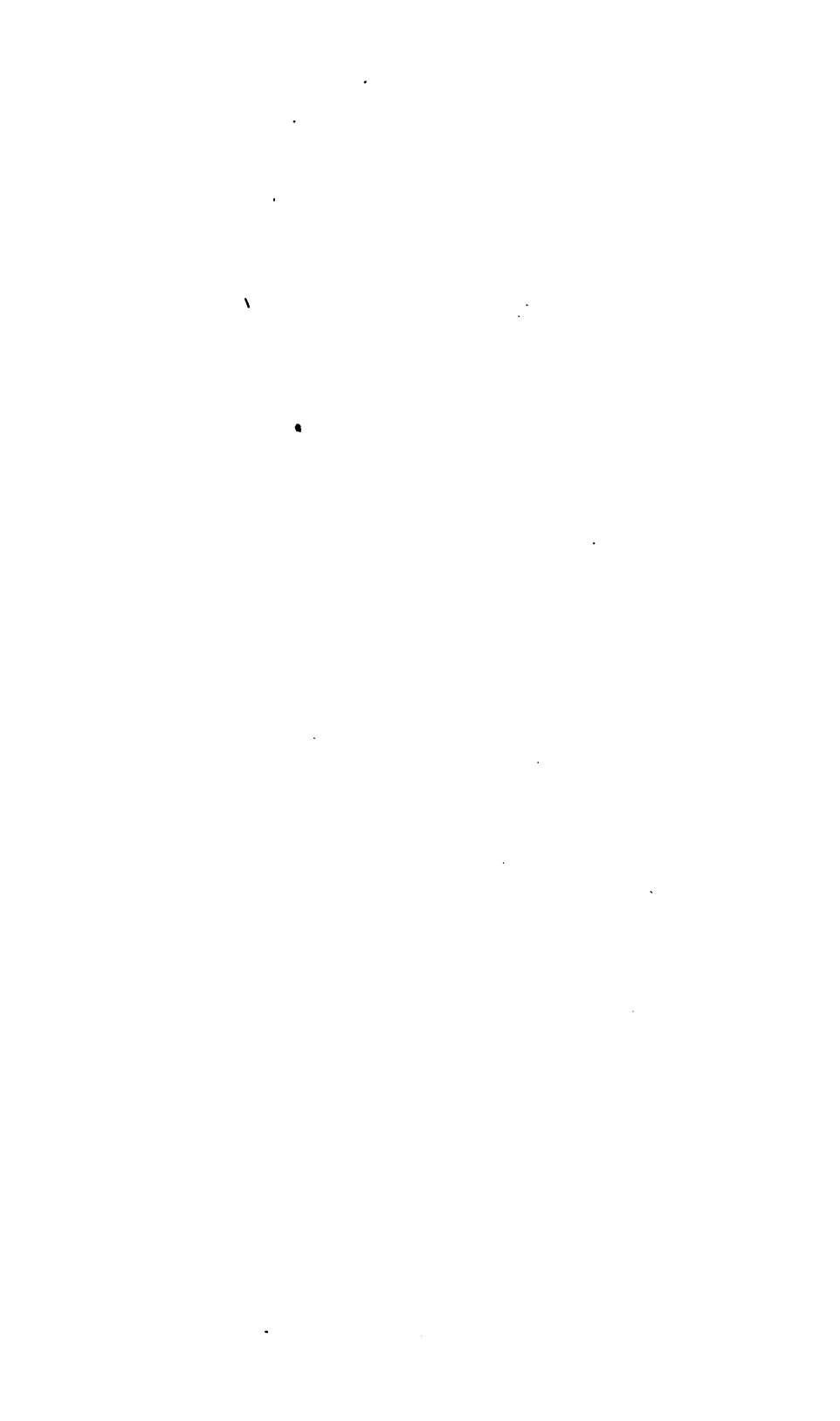
30. What two numbers are those whose difference is 4, and the difference of whose squares is 112 ?

Answer, 12 and 16.

31. What two numbers are those of which the less is to the greater as 2 : 3, and whose product is 6 times their sum ?

Answer, 10 and 15.





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32. What two numbers are those whose product is 108, and whose sum is = twice their difference?

Answer, 6 and 18.

33. Divide the number 48 into two such parts that one part may be three times as much above 20 as the other is less than 20.

Answer, 16 and 32.

34. A rents 25 acres of land at £7 12s. per ann.; one part of the land is rented at 8s. an acre, the remainder at 5s. an acre: how many acres are there of each sort?

Answer, 9 of one kind, and 16 of the other.

35. A gardener is engaged for 36 days on the following conditions; viz. that for every day he laboured he was to receive 2s. 6d.; and for every day he was absent he should forfeit 1s. 6d.; at the end of 36 days he receives £2 18s.: how many days did he work?

Answer, 28.

36. A sum of money is lent at 6 per cent., simple interest. In 10 years the interest amounts to £12 less than the principal: what is the principal?

Answer, £30.

37. A gentleman agrees to give his footman £6 and a livery. At the end of 7 months the servant leaves him, and receives the livery and 50s.: what is the value of the livery?

Answer, 48s.

38. A distributes a pound among 20 persons; to some he gives 6d. a piece, to others 1s. 4d.: how many received 1s. 4d. a-piece?

Answer, 12.

39. Divide 24s. into 24 pieces, consisting only of nine-pences and thirteence halfpennies.

Answer, 8 nine-pennies, and 16 thirteen-pence halfpennies.

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40. A has £100 and B has £48, when they are robbed. Twice as much is taken from A as from B; but A has thrice as much left as B has : how much did each lose?

Answer, A lost £88; B £44.

41. A and B start at the same time from two places, 154 miles distant, and travel till they meet. A travels 8 miles in 2 hours, and B 5 miles in 4 hours : how long, and how far had they travelled before they met?

Let x = number of hours before they meet.

Answer, 56 hours.

42. A sets out on a journey, and travels 7 miles every 5 hours : 8 hours afterwards B starts from the same place, and travels 5 miles every 3 hours till he comes up with A : how long and how far will B travel before he comes up with A?

Answer, 42 hours.

43. A shepherd is plundered by a party of soldiers of half his flock, and half a sheep beside. He is plundered in the same way by a second, a third, and a fourth company, each party plundering him of half the flock he had left, and half a sheep besides; at last he had but 7 sheep left : how many sheep had he at first?

Answer, 127.

44. A buys some eggs at 2 a penny, and the same number at 3 a penny. He afterwards sells the whole at 5 for twopence, and finds he has lost a penny by the bargain : what is the number of eggs he purchased?

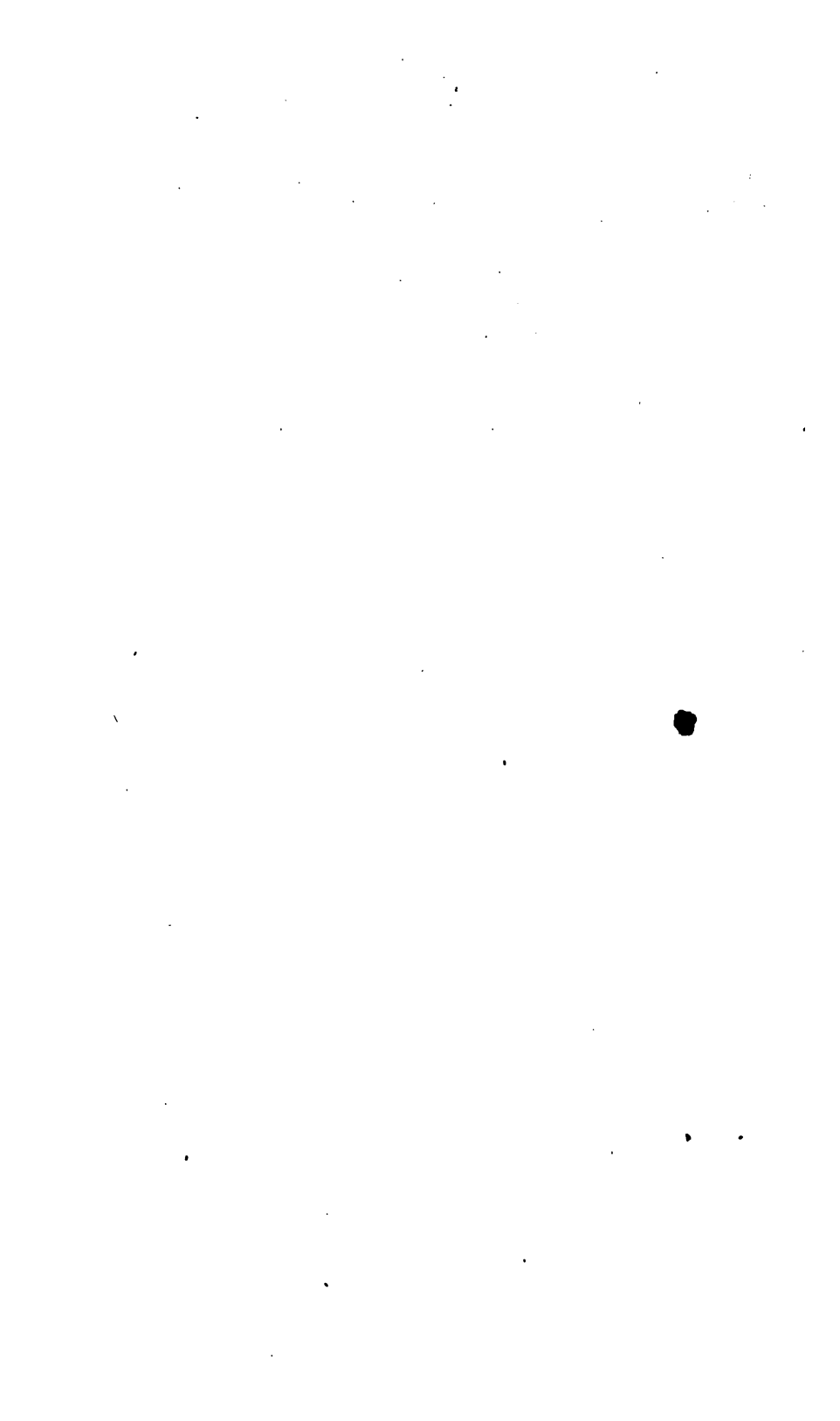
Answer, 60.

45. Divide the number 90 into two such parts that one part may be to the other as 2 : 3.

Answer, the parts are 36 and 54.

46. What number is that which, when added both to 36 and 52, will make the former sum to the latter as 3 : 4?

Answer, 12.



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47. A bookbinder sells me a paper book, containing 48 sheets, for 3s. 4d., and another paper book, containing 75 sheets, for 4s. 10d.; both books were bound at the same price, and both of the same kind of paper : find the price of the binding.

Answer, 8d.

48. What number is that which, when added to 15, 27, and 45, will give three numbers in Geometrical Progression?

Answer, 9.

49. What two numbers are those whose product is 144, and the quotient of the greater divided by the less is 16?

Answer 3 and 28.

50. What two numbers are those, of which the first, added to half the second is = 20 ; and the second added to $\frac{1}{3}$ of the first is also = 20.

Answer, 12 and 16.

51. Six crowns and two dollars are paid for 45s.; also nine crowns and five dollars for 76s.: find the value of a crown and a dollar.

A crown is worth 6s. : 1d.

A dollar is worth 4 : 3

52. Find two numbers, such that $\frac{1}{2}$ the first, added to $\frac{1}{3}$ of the second, may be = 32 ; and $\frac{1}{4}$ of the first added to $\frac{1}{5}$ part of the second, may be = 18.

Answer, 24 and 60.

53. Seven years ago, A was 3 times as old as B ; and seven years hence he will be twice as old : find their present ages.

Answer, 49 and 21.

54. A person has two saddles, one valued at £12, and the other at £2. If he put the better saddle on a certain horse (A), and the worse saddle on another horse (B), A will then be worth twice as much as B ; but if he set the

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better saddle upon B, and the worse upon A, B will then be worth three times as much as A : find the value of the horses ?

A is worth £4.

B..... 6.

55. What is that fraction which becomes $\frac{1}{3}$, when its numerator is increased by unity ; but if unity be added to its denominator, the fraction becomes $\frac{1}{4}$?

Answer, $\frac{4}{13}$.

56. The upper part of a fishing rod is to the lower part as 5 : 7 ; and 9 times the upper part added to 18 times the lower is equal to 11 times the whole rod, and 36 inches over : find the length of each part.

Answer, the upper part was 45 inches,
the lower 63 —

57. A fruiterer expends 2s. 6d. in apples and pears, buying his apples at 4, and his pears at 5, a penny. He finds that he has paid 13d. for half his apples and one-third of his pears : how many of each sort did he buy ?

Answer, 72 apples, and 60 pears.

58. A party at a tavern found, on paying their bill, that if they had been three more in company, they would have had to pay 1s. a-piece less ; and if they had been fewer by two, they would have had to pay 1s. a-piece more than they did : how many were there in company ?

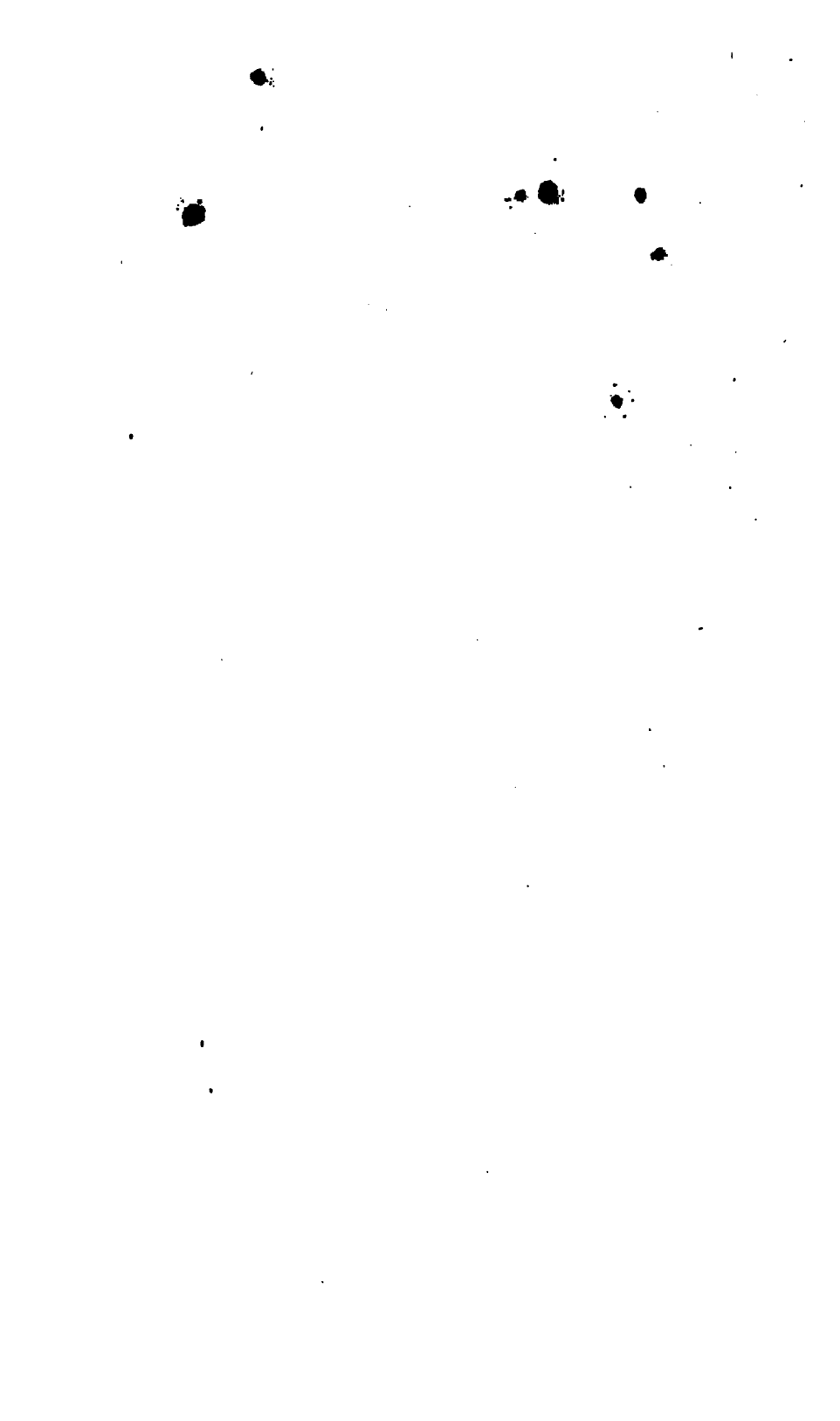
Answer, 12.

59. What two numbers are those whose sum is twice, and whose product is twelve, times their difference ?

Answer, 8 and 24.

60. What two numbers are those whose difference, sum, and product, are the numbers 2, 3, and 5 respectively ?

Answer, 2 and 10.



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61. The difference of two numbers, multiplied into their sum, is $= 5$; and the difference of their squares, multiplied into the sum of their squares, is $= 5$; find the numbers.

Answer, 2 and 3.

62. If B and C give half of their money to A, A will then have £17; if A and C give to B $\frac{1}{2}$ of their money, B will then have £17: if A and B give to C $\frac{1}{2}$ of their money, C will then have £17: how much money has each?

Answer, A has £5; B £11; C £13.

63. A can do a piece of work in 3 weeks; B can do thrice as much in 8 weeks; and C can do five times as much in 12 weeks: in what time can they jointly do it?

Answer, 5 days and 4 hours.

64. A man at play first won as much money as he had in his pocket; afterwards he won the square root of what he then had, and 5s. besides; next he won the square of all he then had; he then finds he has £112 16s.: what had he at first?

Let $\frac{x^2}{2}$ = his money at first.

Answer, 18s.

65. A pack of cards is dealt into 6 heaps; the number of pips on the bottom card, together with the number of cards lying on it, is $=$ to a given number (b) in each heap; after the heaps are dealt there are (d) cards remaining: what is the number of pips on the lowest cards?

Answer, $6(b+1)+d-52$.

66. A person thinks of a number: he then multiplies it by 7, he increases this product by 3; next he divides by 2, then subtracts 4 from the quotient, and the remainder is 15: what is the number he thought of?

Answer, 5.

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67. A boat left Eton (n) days ago, and is rowed (a) miles every day; another boat starts to-day to overtake the first, and goes (b) miles a day: in how many days will the second boat overtake the first?

$$\text{Answer, } \frac{na}{b-a}.$$

68. At what times, and how often, are the hour and minute hands of a clock together in the course of 12 hours?

Answer, they meet at $5\frac{5}{11}$ minutes past 1, and at every multiple of 1 hour $5\frac{5}{11}$ min.; $\therefore$ they meet 11 times in the course of 12 hours.

69. A horse (A) gallops round a course at the rate of (a) miles in a hour; (t) seconds afterwards another horse (B) starts to overtake the first, and gallops (b) miles in a hour: when will B overtake A?

$$\text{Answer, } \frac{at}{b-a}.$$

70. A hare has made 50 leaps before a greyhound starts to catch her: the hare takes 6 leaps to the dog's 5, but 9 of the hare's leaps are equal to 7 of the dog's: how many leaps does the hare make before she is caught?

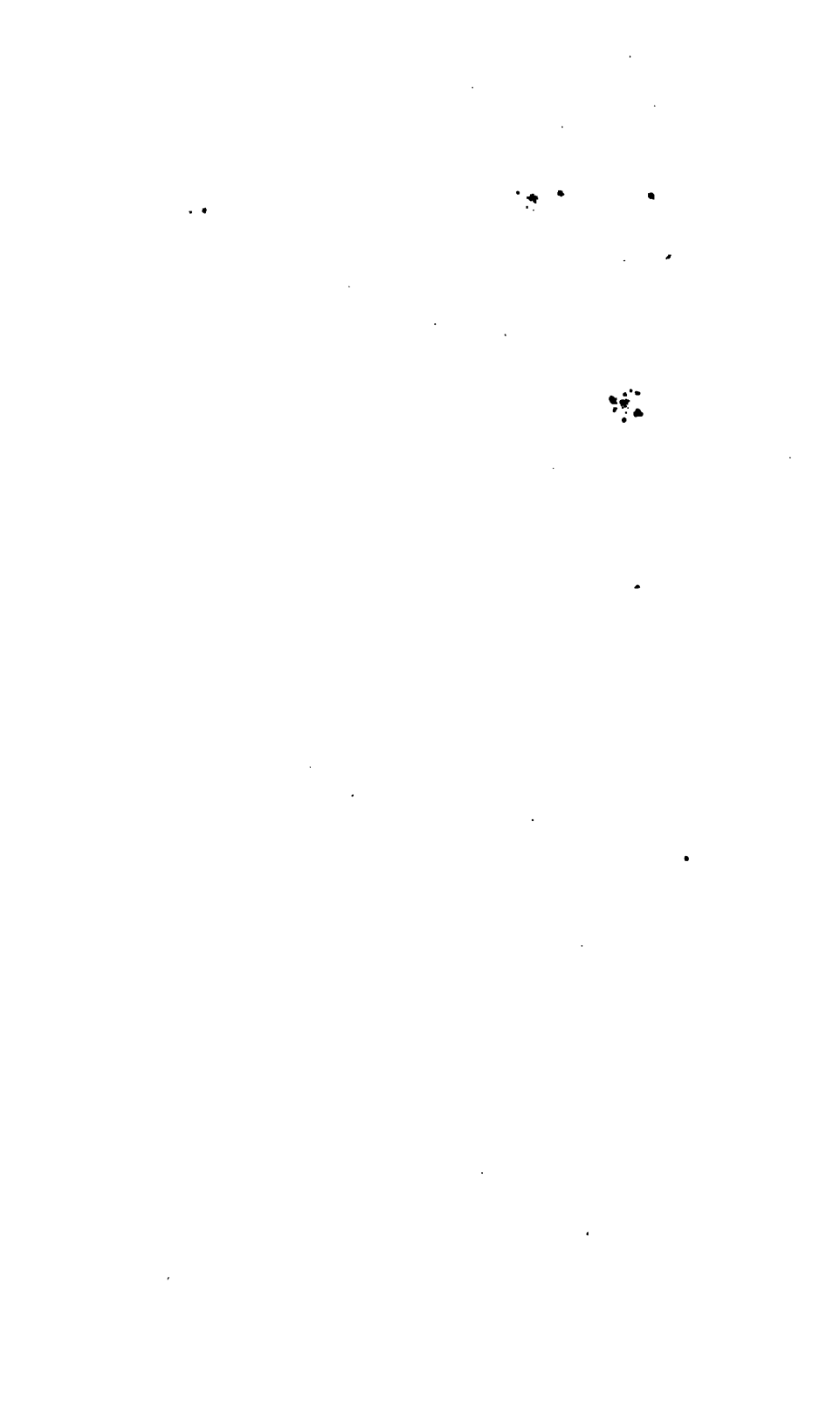
Suppose the hare to take $6x$ leaps, }
 then the dog takes $5x$ leaps. } Answer, 700.

71. A outruns B by (a) feet in a given time; the length of their steps is as $b : c$, the number of steps respectively as $d : e$. What space do they run over?

$$\text{Answer, } \frac{abd}{ce-bd} \text{ and } \frac{ace}{ce-bd}.$$

72. A wine merchant wishes to mix two kinds of wine; one kind costs him (a) pence a quart, the other (b) pence a quart: how much of each kind must he take to make a mixture of (n) quarts, at the rate of (c) pence a quart?

Let x = number of quarts of one kind,
 y = number of quarts of the other.





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Then $ax + by = \overline{x+y} \cdot c$

$$\therefore x \cdot \overline{a-c} = y \cdot \overline{c-b}$$

$$\therefore x : y :: c-b : a-c$$

$$\therefore x : x+y :: c-b : a-b; \text{ but } x+y=n$$

$$\therefore x = n \times \frac{c-b}{a-b}$$

$$\text{Also } x+y : y :: a-b : a-c \quad \therefore y = n \times \frac{a-c}{a-b}$$

73. A wine merchant sells his wine at 7s. a bottle; what quantity of water must he add to 40 bottles of wine that he may sell the mixture at 4s. 6d. a bottle?

Answer, $22\frac{2}{3}$ bottles of water.

74. The ages of two persons are now 30 and 20, i. e. as 3 : 2; in how many years will they be as 5 : 4?

Answer, 20 years.

75. A cistern can be filled by 3 pipes; by the first in $1\frac{1}{2}$ hour, by the second in $3\frac{1}{2}$ hours, by the third in 5 hours: in what time may it be filled by means of all together?

Answer, 48 minutes.

76. Make the latter question general, by supposing the first pipe to fill the cistern in (*a*) hours, the second in (*b*) hours, the third in (*c*) hours.

$$\text{Answer, } \frac{abc}{ab+ac+bc}.$$

77. A countryman sells in the market 4 more than half his eggs; he next sells half the remainder, and 2 over; he next loses half of what is left, and 6 more; and has then 2 eggs remaining: how many had he at first?

Answer, 80.

78. A person left by will, to his eldest son, £100, and a tenth part of the remainder; and the second was to receive £200, and a tenth of the remaining property. The third was to receive £300, and a tenth of the remaining property, and so on. At last it appears that the children were in-

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herited alike : what was the property divided, and what the number of children ?

Let $x\text{£}$ = the property,

Then $100 + \frac{x-100}{10} = \frac{900+x}{10}$ = share of the eldest, $\frac{9}{10}$

$(x-100)-200 = \frac{9x}{10} - 290$; and $\frac{1}{10}$ of this $+ 200 = 200 -$

$29 + \frac{9x}{100} = 171 + \frac{9x}{100}$, share of the second.

Then $171 + \frac{9x}{100} = \frac{900+x}{10}$

$\therefore 171 = \frac{9000+10x}{100} - \frac{9x}{100} = \frac{x+9000}{100}$

$\therefore 17100 - 9000 = x \quad \therefore x = 8100.$

The share of the eldest is $\text{£}900$; and the number of children is 9.

79. A person left by will to his eldest son $(a)\text{£}$, and the n^{th} part of the rest of his property ; to the second son he left $2a\text{£}$, and the n^{th} part of his remaining property ; to the third $3a\text{£}$, and the n^{th} part of his remaining property, and so on ; and at last it appears that his children have inherited alike : find the amount of the property, and number of children.

Answer, $\left\{ \begin{array}{l} \overline{n-1}^2 a \text{ is the property, and} \\ (n-1) \text{ the number of children.} \end{array} \right.$

80. A person has a certain number of shillings, which he wishes to arrange in the form of a square. His first square was so small that he had 130s. remaining over ; his second square admitted 3s. more on one side, and he had then only 31 remaining over : how many shillings had he ?

Answer, 355.



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81. A merchant mixes 3 quarts of one wine with 5 quarts of another; and can then sell the mixture for 17*s.* 1*d.* the quart. But if he mix $3\frac{1}{2}$ quarts of the first with $7\frac{1}{2}$ quarts of the second, he can sell it for 16*s.* 8*d.* the quart: what is the price of each wine?

Let x = price of the first kind,

y = price of the second,

$$\therefore 3x + 5y = 205$$

$$\frac{15x}{4} + \frac{15y}{2} = 200 \text{ \&c.}$$

82. A person sells one kind of wine at 15*s.* a gallon, and another kind at 22*s.* a gallon. In what proportions must they be mixed, that the compound may sell at 18*s.* a gallon?

Let (x) and (y) be the required gallons of each,

$$\text{Then } 15x + 22y = 18(x + y)$$

$$\therefore 4y = 3x, \text{ or } x : y :: 4 : 3$$

83. How must wines at 17*s.* 18*s.* and 22*s.* a gallon be mixed so as to sell at 20*s.* a gallon?

Let there be (x) gals. at 17*s.* (y) gals. at 18*s.* (z) gals. at 22*s.*

$$17x + 22z = 20(x + z) \quad \therefore 2z = 3x \quad \therefore x : z :: 2 : 3$$

$$\text{Again, } 18y + 22z = 20(y + z) \quad \therefore y = z.$$

Hence there must be equal quantities of wines at 18*s.* and 22*s.*, and for every 2 gals. at 17*s.* there must be 3 gals. at 22*s.*; so that 9 gals. of the mixture contains 2 gals. at 17*s.* 2 gals. at 18*s.* and 5 gals. at 22*s.*

84. Upon this principle is founded the rule of alligation in arithmetic:

$$\begin{array}{l} \text{Thus } \left. \begin{array}{l} 17 \\ 18 \\ 20 \\ 22 \end{array} \right\} \begin{array}{l} 2 \text{ at } 17 \\ 2 \text{ at } 18 \\ 3 + 2 \text{ at } 22. \end{array} \end{array}$$

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85. Two cubic inches of one metal (*A*) weigh 1 lb. ; and 5 cubic inches of another metal (*B*) weigh 1 lb. : in what proportion can they be mixed, that 3 cubic inches of the compound (*C*) may weigh 1 lb.

Let x represent the number of lbs of A, and y the number of lbs of B; then $2x$ will express the cubic inches of A, and $5y$ the cubic inches of B,

Then $2x + 5y = 3(x + y) \therefore 2y = x \therefore x : y :: 2 : 1$
2lbs of A to 1lb of B.

86. A mass (A) of metal weighs (n) lbs, and loses (a) lbs when immersed in water; another mass (B) of the same weight loses (b) lbs when immersed in water: in what proportions may these metals be mixed so that (n)lbs of the compound may lose (c) lbs in water?

It is manifest that a , b , and c represent the quantities of fluid displaced, and $\therefore$ may represent the relative magnitudes of the masses.

Suppose the compound consists of (x) lbs of A and (y) lbs of B , then, as in the last example,

$$ax+by=c(x+y) \quad \therefore x : y :: b-c : c-a$$

$$\therefore x : x+y :: b-c : b-a$$

or $x : n :: b - c : b - a$

$$\therefore x = n \times \frac{b-c}{b-a}$$

and $y = n \times \frac{c-a}{b-c}$

87. Suppose Hiero's crown, when immersed in water, to have lost $\cdot 64$ lbs; and that a gold mass of the same weight would lose $\cdot 52$ lbs, and a silver mass of the same weight would lose $\cdot 92$ lbs in water: find the relative quantities of gold and silver in the crown.

The relative magnitudes of the masses are $\cdot 52$, $\cdot 64$, and $\cdot 92$; or 52, 64, and 92.





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Let the gold and silver in the crown be (x) lbs and (y) lbs respectively,

$$52x + 92y = 64(x + y) \quad \therefore x : y :: 7 : 3$$

For 7 lbs of gold there were 3 of silver.

88. A confectioner sells cakes at as many pence a piece as the number of cakes he has, and on giving back $2d.$ he finds that if he had had one more cake to sell on the same terms, and had given back $3d.$ he would have received $3s. 4d.$ more from his bargain: how many cakes did he sell?

If he sold (x) cakes,

$$(x+1)^2 - 3 = x^2 - 2 + 40 \text{ &c.} \quad x = 20.$$

89. A farmer was robbed of one-fifth part of a basket of eggs: he next finds that one-eighth of the original number + 6 are broken, and that he has only half his original number + 36 remaining entire: how many eggs had he at first?

Answer, 240.

90. Find three numbers, such that $\frac{1}{3}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third shall be 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third shall be 38.

Answer, 24, 60, and 120.

91. A sum of money is divided among 4 people, A, B, C, and D; A's share was half the sum of the other three; B's share was one-third the sum of the other three; C's share was one-fourth the sum of the other three; and A's share exceeded D's by £14: find the share of each.

Answer, £26, £30, and £40.

92. Find four numbers, such that the first together with half the second may be (a) ; the second together with one-third of the third may be (b) ; the third with one-fourth of the fourth may be (c) ; and the fourth together with one-fifth of the first may be (d) .

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93. Divide 90 into four such parts that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same.

Answer, 16, 25, 7, and 42.

94. What number is that whose square exceeds its simple power by 306 ?

Answer, 18.

95. What numbers are those whose sum is 41, and the sum of whose squares is 901 ?

Answer, 26 and 15.

96. What numbers are those whose difference is 8, and the sum of whose squares is 544 ?

Answer, 12 and 20.

97. What numbers are those whose product is 255, and the sum of whose squares is 514 ?

Answer, 15 and 17.

98. Into what two parts must 39 be divided so that the sum of their cubes may be 17199 ?

Answer, 15 and 24.

99. A person buys some pieces of cloth at equal prices for £60 ; had he got 3 pieces more for the same sum, each piece would have cost 20s. less : how many pieces did he buy ?

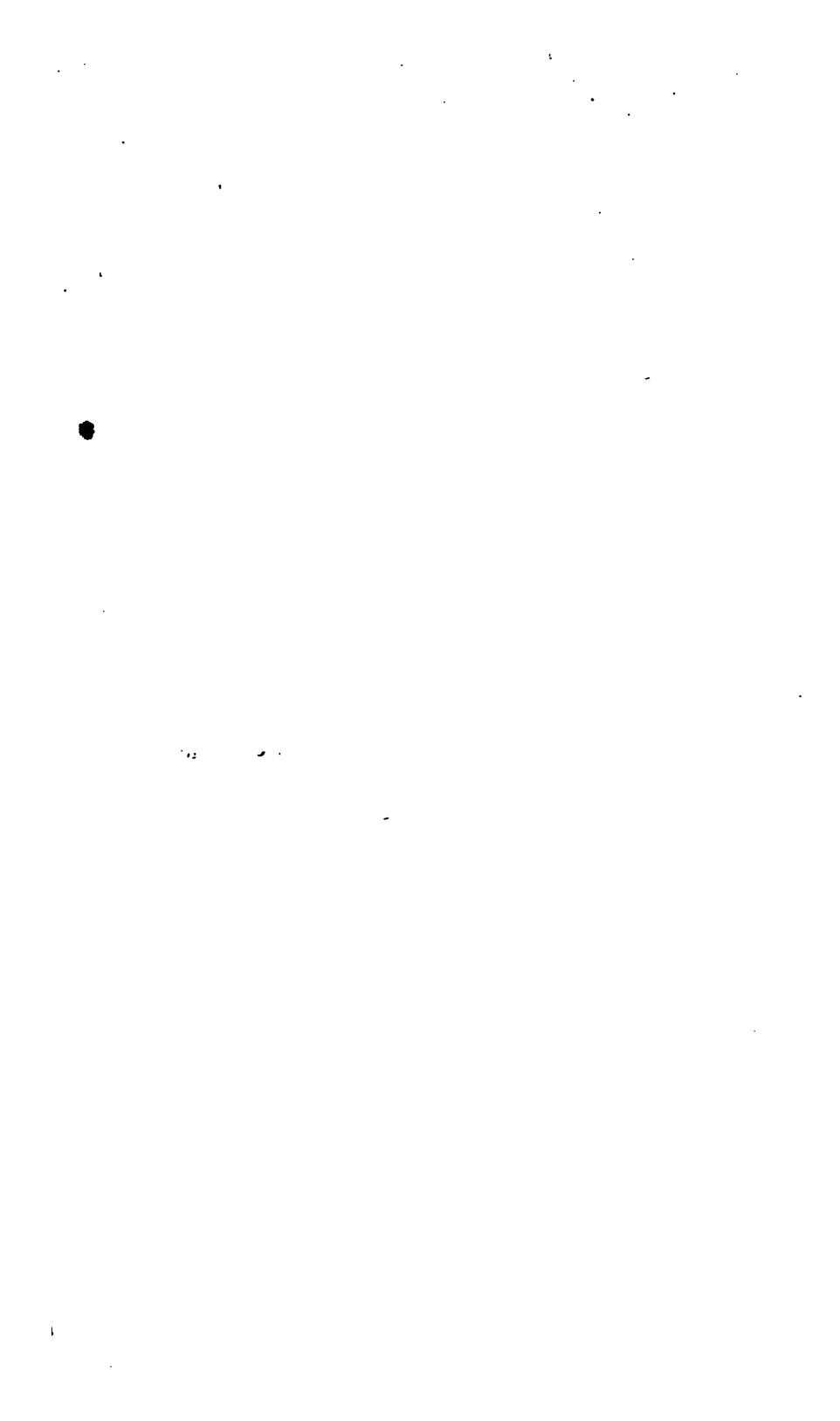
Answer, 12.

100. £36 was to have been divided among a company of paupers ; six of them were missing, consequently each receives 10s. 8d. more than he otherwise would have done : how many were there ?

Answer, 54.

101. Find a number (x), such that if a given number (c) be divided by (x), and be also divided by ($x+a$), the difference of the quotients is (d).

Answer, $-\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + \frac{ac}{d}}$



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102. Divide 36 into three such parts that the second may exceed the first by 4; and that the sum of their squares may be 464.

Answer, 8, 12, and 16.

103. Divide 100 into two such parts that their product and the difference of their squares may be equal to each other.

Answer, $x=38.19658$.

104. Having given the sum of two numbers, and the sum of their $(n)^{\text{th}}$ powers, find the numbers.

Let $a+x$ =greater number,

and $a-x$ =the less,

If $n=3$; $\overline{a+x}^3 + \overline{a-x}^3 = b$; &c.

If $n=4$; $\overline{a+x}^4 + \overline{a-x}^4 = b$; &c.

If $n=5$; $\overline{a+x}^5 + \overline{a-x}^5 = b$; &c.

105. Find two numbers, such that their sum, their product, and the difference of their squares may be equal each other.

Answer, $\frac{1}{2} + \sqrt{\frac{5}{4}}$ and $\frac{3}{2} + \sqrt{\frac{5}{4}}$

106. Find three numbers in arithmetical progression, such that the sum of their squares shall be 1232, and the square of the mean shall exceed the product of the two extremes by 16.

Answer, 16, 20, and 24.

107. Divide 134 into three such parts that once the first, twice the second, and three times the third may together be 278; and that the sum of their squares may be 6036.

Answer, 40, 44, and 50.

108. There are four numbers in arithmetical progression, the product of the extremes is 3250, and the product of the means is 3300: find the numbers.

Answer, 50, 55, 60, and 65.

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109. The sum of three numbers in arithmetical progression is 30, and the sum of their squares is 308 : find the numbers.

Answer, 8, 10, and 12.

110. What number is that which exceeds its square root by $48\frac{3}{4}$?

Answer, $56\frac{1}{4}$.

111. Find two numbers, such that their difference added to the difference of their squares shall be 150 ; and their sum added to the sum of their squares shall be 330.

Answer, 9 and 15.

112. Find three numbers in geometrical progression, such that their sum shall be 14, and their continued product 64.

Answer, 2, 4, and 8.

113. Find two numbers whose difference multiplied by the difference of their squares shall be 160 ; and whose sum multiplied by the sum of their squares shall be 580.

Answer, 3 and 7.

114. Find two numbers, such that their sum and product may together amount to 34 ; and the sum of their squares may exceed the sum of the numbers by 42.

Answer, 4 and 6.

115. Having given the sum (S) of two numbers, and their product (P), find the sum of their squares, cubes, and biquadrates.

$$\text{Since } x+y=S$$

$$\text{and } xy=P$$

$$\overline{x+y}^2=S^2$$

$$2xy=2P$$

$$x^2+y^2=S^2-2P$$

$$\text{Again, } x+y=S$$



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$$\therefore x^3 + xy \times (x + y) + y^3 = S^3 - 2SP$$

$$\text{or } x^3 + P \times S + y^3 = S^3 - 2SP$$

$$\therefore x^3 + y^3 = S^3 - 3SP$$

$$\text{Again, } x + y = S$$

$$\therefore x^4 + xy(x^2 + y^2) + y^4 = S^4 - 3S^2P$$

$$\text{or } x^4 + P \times (S^2 - 2P) + y^4 = S^4 - 3S^2P$$

$$\therefore x^4 + y^4 = S^4 - 4S^2P + 2P^2.$$

116. The sum of two numbers is 12, the first divided by the second + the second divided by the first is equal to $\frac{5}{2}$: find the numbers.

$$\text{Let } 6 + x = \text{greater number,}$$

$$\therefore 6 - x = \text{less number,}$$

$$\therefore \frac{6+x}{6-x} + \frac{6-x}{6+x} = \frac{5}{2} \text{ \&c.}$$

$$\text{If the equation be proposed in this form } \begin{cases} \frac{x}{y} + \frac{y}{x} = \frac{5}{2} \\ x + y = 12 \end{cases}$$

$$\text{Let } x = z + v$$

$$y = z - v$$

$$\therefore x + y = 2z = 12 \quad \therefore z = 6$$

$$\text{Also } xy = z^2 - v^2 = 36 - v^2$$

$$\text{Now } x^2 + y^2 = \frac{5}{2} \times xy$$

$$\therefore 2(z^2 + v^2) = \frac{5}{2}(z^2 - v^2)$$

$$\therefore 9v^2 = z^2 \quad \therefore 3v = z = 6 \quad \therefore v = 2$$

$$\therefore x = 8; y = 4$$

$$\text{Given } \frac{x^2}{y} + \frac{y^2}{x} = \frac{65}{2} \left. \vphantom{\frac{x^2}{y} + \frac{y^2}{x} = \frac{65}{2}} \right\} \text{ to find } x \text{ and } y.$$

$$\text{and } x + y = 10$$

$$\text{Assume, as before, } x = z + v$$

$$y = z - v$$

$$\text{and we find } z = 5, \text{ and } v = 3; \quad \therefore x = 8$$

$$y = 2.$$

In like manner we can find the values of x and y in the following equations :

P.
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$$\left. \begin{array}{l} \frac{x^3}{y} + \frac{y^3}{x} = a \\ x + y = b \end{array} \right\}$$

$$\left. \begin{array}{l} \frac{x^4}{y} + \frac{y^4}{x} = a \\ x + y = b \end{array} \right\}$$

This mode of substitution always applies when the unknown quantities are similarly involved.

Divide $99\frac{9}{10}$ into two such parts that,

the $1^{th} + 5\frac{1}{2} =$ the $2^{nd} - 2\frac{3}{7}$

Ans $\left\{ \begin{array}{l} 1^{th} = 45\frac{69}{70} \\ 2^{nd} = 53\frac{32}{35} \end{array} \right.$

THE END





